NET 2022 Power Round

Advanced Division: Microeconomics

April 2022

Instructions

This is the **microeconomics portion** of the Wilson division of the 2022 Northwestern Economics Tournament Power Round. There are three questions of *unequal* weight, accounting for a weighted *half* your score for the Power Round. You are encouraged to work together on these questions. Answer each question as clearly and succinctly as possible. You may write on a blank sheet of paper where you *clearly indicate* where your answer to each part is. It will be useful to note that a question's point value is *not* informative of its difficulty; to ensure a fair test, some (longer) easy questions are worth more points, while some (shorter) hard questions are worth less points, and vice versa. If you are unsure of your answer, take your best guess: there is no penalty for incorrect answers. If you find yourself stuck on a question, skip it and return to it at the end if necessary. It is recommended you spend approximately seventy five (75) minutes of the total exam time on this portion. Remember, we do *not* share your answers or scores with Northwestern admissions, nor do we keep them for ourselves. You are not expected to know how to answer each question on the exam; rather, this test is designed to assess your economic and formal reasoning skills. Have fun, and good luck!

Problem 1: Oligopoly Types (24 Points)

In this question, we explore the mathematics behind three different models of market competition with two firms (Cournot, Collusion, and Stackelberg), comparing the resulting competitive equilibria.
Suppose two profit-maximizing firms operate in the same market and sell identical products. The products have a constant marginal cost of 5 per unit and there are zero fixed costs. The market price depends on the total quantity of goods in the market. The inverse demand function is

$$P = 149 - Q$$

where we take $Q = q_1 + q_2$ (the quantities produced by firm 1 and 2 respectively). This is a duopoly, where a firm's decision on how much to produce affect the profits of both firms.

Part A (2) Assume that there is no collusion and that the firms opperate independently. Also firms choose their quantities at the same time, such as on the first day of the year. We call this the Cournot duopoly. Write the profit functions for each firm in terms of $q_1$and q_2.

Part B (3) By maximizing the profit functions, calculate the quantity firm 1 produces in terms of the quantity firm 2 produces. Then calculate firm 2's quantity produced in terms of firm 1's quantity produced. (Hint: What are the first order conditions?)

Part C (3) Solve the system of equations you wrote down to find the equilibrium quantity for each firm and the equilibrium price in the market. Finally, find the profit for both firms using the price and quantity you found.

Part D (2) Now assume the firms completely collude and act as one single firm in the market. Remembering that there are no other firms in the market, give the name for this market structure and the profit function for this new firm.

Part E (4) Find the equilibrium quantity produced by the new firm, the equilibrium price in the market, and the profit of the firm.

Part F (2) Now consider the case where the firms are again operating independently as two firms. In this case, however, firm 1 will choose their quantity *before* firm 2 does, giving firm 2 a clear chance to "react" to firm 1's choice. This is called Stackelberg competition. Describe what you expect will happen to the profits of these two firms compared to the Cournot duopoly state.

Part G (5) Prove what you stated above by finding the new equilibrium quantities, price, and profits. As a hint, you can use a technique called backwards induction: firm 2 will choose their quantity to maximize their profits conditional on what firm 1 has already done. Firm 1 knows this, so, given that firm 2 is going to take this reaction, firm 1 will choose their quantity to maximize their profits conditional on what firm 2's quantity *will* be after firm 1 makes their choice.

Part H (3) Compare the three market equilibria you just found in parts C, E, and G. Rank them in order of the combined profits of the firms and then in terms of consumer surplus.

Problem 2: Bargaining, Fairness, & Money on the Table (24 Points)

This problem considers Rubinstein's Alternating-Offers Bargaining Game and investigates the (in)finite horizon Nash Equilibrium[1]*. This problem does not require any knowledge of calculus or prerequisite knowledge outside of some basic game theory, e.g. the definition of Nash equilibrium.*

Part A (2) Consider the following scenario: there are two people, R and G, there is one dollar on the table. R can pick any way to split the dollar and offer some $\theta \in [0, 1]$ of the dollar to G. G can accept, in which case they walk away with that offer, or reject, in which case both players get nothing. Assume if players are indifferent between accepting and rejecting, they accept. What is the unique best offer R can make G?

Part B (7) Now assume there are two periods of the game, and that players *discount the future*: that is, a dollar today is only worth δ tomorrow (you can think about this as the cost of time). Moreover, assume that bargaining power *alternates*:
Part 1 (1) Suppose that the offer on the table in the first period to Player G is 0.7. How much would G have to receive in the second period to (weakly) prefer waiting? (Write your answer as a function of δ).
Part 2 (2) Bargaining power is alternating. Using your answer in Part (A), what will G offer R if the game is allowed to proceed to the second stage?
Part 3 (2) Assume that R is very intelligent and knows what will happen if G rejects their offer. What should R offer G in the first period to make G indifferent between acceptance and rejection?
Part 4 (2) Explain why the strategy consisting of R offering the value from Part (B3) and G accepting is a Nash equilibrium in this game. (Hint: Consider your answer in Part (A)).

Part C (3) Find the Nash equilibrium of the game when there are three total periods of bargaining and power still alternates (so R offers in the last round). *(Hint: Your answer should be a function of δ. Use the outline as in Part (B).)*

Part D (4) Following the answer in Parts (B) and (C), suppose in a one-period game, if G rejects R's offer, the surplus split will be $(x, 1-x)$ (assigned to R and G respectively) for some $0 \leq x \leq 1$. What will be the equilibrium of this one period game? (Your answer will be a function of x and δ). Given this formula, can you calculate the equilibrium offers in 4 periods? 5? State what they are and explain why. *(Hint: It may be useful to create a table that keeps track of the offerer, payoffs from future (discounted) periods, and offers in each time period. Work backwards from the last period of the game: this is called backwards induction.)*

Part E (4) A linear difference equation with initial condition y_0 is a function of form $y_{t+1} = a + by_t$ for real numbers a, b. You may take as a mathematical fact that, given y_0, we can write the t-th term of the difference equation, y_t, as

$$y_t = \left(\sum_{k=0}^{t-1} a(b)^k \right) + b^t y_0$$

With this fact in mind, what is the Nash equilibrium of the repeated offer game in n periods? It is sufficient for you to specify an initial offer at time $t = 0$. *(Hint: You should apply the linear difference equation to the offer being made, regardless of the specific person doing the offering. Can you see a pattern? Apply your answers in Parts (D) and (B/C).)*

Part F (3) Conjecture as to what happens as $n \to \infty$. How many periods does it take for the game to end? What does the sequence of offers look like in equilibrium?

Part G (1) Conjecture, in a few sentences, a situation that could be modeled by the alternating-offer game we described above.

Bonus (3) Assume the game remains as described, except now players have *distinct* discount factors δ_R, δ_G, respectively. Find the Nash equilibrium of the infinite-horizon game.

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[1] The proof, due to Rubinstein, establishes our argument is, in fact, the unique subgame perfect equilibrium.

Problem 3: Winners Win: When More is Better (25 Points)

In this problem we investigate particularly influential work by nobel laureate Paul Milgrom and coauthor Chris Shannon in 1994, and in a more primitive form first developed by Donald Topkis in the 1980s.

Consider an agent facing a decision where they choose an element x in a set X to maximize their payoff, but where their payoff depends on some state of the world (a *type parameter*), denoted by some t in a set T. Let both x and t be real numbers and ordered in an intuitive way, so that if $x_1 > x_2$, we can think of x_1 as a "higher" or "more extreme" action than x_2, while if $t_1 > t_2$, then the state of the world t_1 is likewise "higher", or "more extreme" than t_2. The agent's choice maximizes their utility function; their optimal decision x^* at type t is given by

$$x^*(t) = \underset{x \in X}{\operatorname{argmax}}\, u(x,t)$$

that is, they pick x^* to maximize their utility $u(x,t)$ given the state of the world is t, where $u(x,t) : X \times T \to \mathbb{R}$ is a real-valued function. We are interested in how $x^*(t)$ changes as t changes.

Part A (3) Suppose that $u(x,t) = -t(x-t)^2$ where $x \in X$ is any real number and $t \in T$ is (weakly) positive: $t \geq 0$ always. Either through solid reasoning or calculus, find $x^*(t)$ for each t. How does this value change with t?

Part B (2) It turns out that there is really nothing special about the family of inverse parabolas given in Part (A) and the monotoncity of $x^*(t)$. Instead, the relevant condition is that the utility function satisfies *increasing differences*:

$$u(x', t') - u(x, t') \geq u(x', t) - u(x, t)$$

for all $x' > x$ and $t' > t$. Interpret and explain this condition in the language of economics, using your intuition.

Part C (7) Recall in Part (A) that $x^*(t)$ varied regularly with changes in t. This relationship is a feature of *monotone comparative statics*: the optimal decision an individual makes is monotone in changes in some model parameter. Follow the below outline to show that increasing differences is a sufficient condition for a consistent monotone comparative statics result. Throughout, assume that $u : X \times T \to \mathbb{R}$ is increasing differences and let $x^*(t) = \operatorname{argmax}_{x \in X} u(x,t)$ be single-valued. Now, we want to show that $x^*(t)$ must be increasing in t.

(1) *(1 point)* Assume not; that is, $x^*(t)$ is not weakly increasing in t. Complete the following statement: there exists $t_1 < t_2$ such that $x^*(t_1)$ is (greater than, lesser than, equal to) $x^*(t_2)$. The assumption for contradiction is a (strict, weak) inequality.

(2) *(2 point)* Write out what it means for each argument to be optimal at t_1 and t_2 (hint: $x^*(t)$ must give the highest utility out of *all* the x at t; write this as an inequality).

(3) *(3 points)* Using the inequalities from (1) and (2), show that $u(x,t)$ violates increasing differences.

(4) *(1 point)* Explain why the above steps were sufficient to conclude your proof.

Part D (4) Often, it is difficult to show that a function $u(x,t)$ is increasing differences. When $u(x,t)$ is twice differentiable in both its arguments, though, the following are equivalent:

(1) $u(x,t)$ satisfies increasing differences in (x,t).

(2) The derivative of u with respect to x, $\frac{\partial u}{\partial x}(x,t)$ is an increasing function in t.

Using the above equivalence, (1) show that the original quadratic loss functions $u(x,t) = -t(x-t)^2$ is increasing differences, and (2) explain intuitively why you would expect these two conditions to be equivalent from an economic perspective.

Part E (2) Explain why, for twice-differentiable functions, if $u_{xt}(x,t) = \frac{\partial}{\partial t}\left(\frac{\partial u(x,t)}{\partial x}\right) \geq 0$, then u is increasing differences. *(Hint: Use Part (D)).*

Part F (4) Consider now a monopoly which faces an inverse demand function $p(q)$ and constant marginal costs c, so that they seek to maximize profits, which are given by

$$\pi(q, c) = p(q) \cdot q - cq$$

Without putting any assumptions on the inverse demand function $p(q)$, show that if the marginal cost increases, the firm will decrease its production. *(Hint: The type space is "flexible": can you pick a definition of the type so that π is increasing differences? Note costs can be seen as negative revenue.)*

Part G (3) Consider now instead a similar scenario, except for every unit produced the firm gets a government subsidy equivalent to $\frac{1}{4}cq$ (so they get a higher subsidy if they produce more), while the inverse demand function is $p(q) = 100 - 5q$. Then, the objective function is

$$\pi(q, c) = (100 - 5q) \cdot q - \left(c - \frac{1}{4}cq\right) \cdot q$$

Find two costs, c_1 and c_2, such that quantity produced at c_1 is higher than quantity produced at c_2, but $c_1 < c_2$. Is this consistent with the monotone comparative statics result? Why or why not?

Bonus (3) The increasing differences result is *cardinal*: it is not preserved under monotone transformation, so it is a property of the specific utility function instead of the underlying preferences[2]. Alternatively, there exists a (weaker) ordinal result: say that $u(x, t)$ satisfies *single crossing* if, for all $t' > t$, $u(x', t) - u(x, t) \geq 0$ implies $u(x', t') - u(x, t') \geq 0$ whenever $x' > x$. Using a similar outline as that in Part (C), prove the *Milgrom-Shannon Univariate Monotone Comparative Statics theorem*: If $u(x, t)$ is single-crossing and has a unique maximum in x for each t, then $x^*(t)$ is increasing in t.

[2] This is especially apparent by comparing Parts (F) and (G), which in some sense represent the same function