Optimal auction design with common values: An informationally-robust approach*

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Abstract

A Seller has a single unit of a good to sell to a group of bidders. The good is costly to produce, and the bidders have a pure common value that may be higher or lower than the production cost. The value is drawn from a prior distribution that is commonly known. The Seller does not know the bidders' beliefs about the value and evaluates each auction mechanism according to the lowest expected profit across all Bayes Nash equilibria and across all common-prior information structures that are consistent with the known value distribution. We construct an optimal auction for such a Seller. The optimal auction has a relatively simple structure, in which bidders send one-dimensional bids, the aggregate allocation is a function of the aggregate bid, and individual allocations are proportional to bids. The accompanying transfers solve a system of differential equations that aligns the Seller's profit with the bidders' local incentives. We report a number of additional properties of the maxmin mechanisms, including what happens as the number of bidders grows large and robustness with respect to the prior on the value.

Keywords: Mechanism design, information design, optimal auctions, profit maximization, common value, information structure, maxmin, Bayes correlated equilibrium, direct mechanism.

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1 Introduction

1.1 Background and motivation

We study the design of profit-maximizing auctions when the bidders have a pure common value for a good being sold, but partial and differential information about that value. The common value model is a natural approximation for many real world markets, such as those for natural resources or financial assets, where to a first order all bidders have the same preferences about the market value of the resource or the future cash flows of the asset.

Common-value auctions have been studied almost since the beginning of auction theory. And yet relatively little is known about optimal auctions. When bidders' signals are independent and one-dimensional, Bulow and Klemperer (1996) have argued that English-like auctions are optimal under a condition that signals associated with higher expected values are not too precise. In the perhaps more natural case where signals are correlated through the common value, such as in the mineral rights model, McAfee, McMillan, and Reny (1989) and McAfee and Reny (1992) construct mechanisms that extract all of the surplus by having the bidders bet on other bidders' information. While the full-surplus extracting auctions are theoretically interesting, there are a number of reasons why they may not be practically useful, including that the designer may not know exactly how information is correlated, and the optimal auction may be too complicated for bidders to use.

This discussion points to some conceptual issues in optimal auction design. First, the optimal auction varies widely with the model of bidders' information, e.g., whether and how signals are correlated. Second, it is hard to determine, either through measurement or introspection, which model of information is empirically relevant. Third, relatively little is known about how optimal auctions would fare if the information model is misspecified. Note that these issues also arise in the private-value setting, but at least there the independent private value model has been broadly accepted as a useful benchmark. In common-value auctions, there is no comparably canonical model.

To address these issues, we model a Seller who knows the distribution of the common value, but faces ambiguity about the information that bidders have about the value. The latter is modeled as a common-prior *information structure*. The Seller is concerned about model misspecification, and evaluates each auction design according to its lowest expected profit across all information structures and Bayes Nash equilibria. We refer to this minimum as the auction's *profit guarantee*. The problem is to identify the auction that provides the largest such profit guarantee.

1.2 Main results

Our main result is to explicitly construct an auction mechanism that maximizes the profit guarantee. When the number of bidders is large, the guarantee is approximately the entire ex ante gains from trade, i.e., the expectation of the value under the prior minus the cost of production (or zero if the expected value is less than the cost). While we do not formalize this idea, the guarantee seems to be a substantial share of surplus even when the number of bidders is small. For example, when there are two bidders and the value is standard uniform and there is zero cost of production, the mechanism we construct guarantees the Seller at least 56 percent of the total surplus as profit.

Our solution actually consists of both a maxmin mechanism and a minmax information structure. The mechanism provides the optimal profit guarantee, and the information structure certifies that this guarantee is unimprovable since no mechanism can do better at the minmax information structure. In our analysis, we first derive the minmax information structure using the heuristic that the Seller should be indifferent between a wide range of mechanisms. The mechanism is then constructed to be an optimal direct mechanism on the minmax information structure with the additional feature that profit must be weakly higher in any equilibrium on any information structure. Thus, the messages in the maxmin mechanism are "normalized" to be signals in the minmax information structure. We refer to this structure as the *double revelation principle*: The maxmin mechanism is a profit-maximizing direct mechanism on the minmax information structure, and the minmax information structure is a profit-minimizing correlated equilibrium on the maxmin mechanism. The existence of a solution of this form is a non-trivial result, and it does not follow from the standard revelation arguments.

The requirement that the profit be minimized at the minmax information structure reduces to a pair of differential equations and an integral equation involving the mechanism's allocation and transfer rules. The first differential equation pins down the divergence of the allocation rule, i.e., the sum of the partial derivatives of each bidder's allocation probability with respect to their own message. We refer to this as the *aggregate allocation sensitivity*. The solution to this equation has the following form: The aggregate probability of the good being sold is a function of the aggregate message, i.e., the sum of the messages, and conditional on this aggregate supply, the good is allocated to each bidder with a likelihood that is proportional to their message.

In benchmark cases, the aggregate supply is linearly increasing in the aggregate message until it hits 1 and stays constant as 1 thereafter. An interpretation is that messages are "demands" for a quantity of the good. The demands are completely filled when the aggregate demand is less than the available supply, and otherwise the good is rationed in a proportional manner.

The second differential equation links ex post profit (which depends on the sum of the transfers) to the bidders' local incentives (which depends on the divergence of the allocation rule and the divergence of the transfer rule). We refer to this relationship as *profit-incentive alignment* (PIA). The maxmin transfer rule solves PIA, subject to an additional integral equation that is necessary for the transfers to be bounded when messages are large. This transversality condition rules out pathological solutions for which equilibria do not exist on any information structure.

One can view our solution as a saddle point of a zero-sum game between the Seller, who chooses the mechanism to maximize profit, and adversarial Nature, who chooses the information structure to minimize profit. A subtlety in modeling and solving this game is that for a fixed mechanism and information structure, there can be more than one equilibrium with different levels of profit. One might therefore be concerned that the solution depends on the equilibrium selection rule. For the solution we identify, however, neither the Seller nor Nature can move profit in their preferred direction by changing the mechanism or information structure, respectively, even if the deviating party can also select the equilibrium. In our view, this is a surprising and normatively desirable feature of the solution.

We highlight this result by defining and using a new solution concept that we term a *strong maxmin solution*, which builds in the requirement that the profit guarantee must not depend on how we select an equilibrium. Theorem 1 shows by construction that a strong maxmin exists. Theorem 3 goes on to show that the profit guarantee for a strong maxmin solution is unique among solutions that are sufficiently well-behaved. After Theorem 3, we relate our solution back to the original maxmin motivation by exhibiting a collection of maxmin mechanism design and minmax information design problems with a wide range of equilibrium selection rules that are all solved by the strong maxmin solution.

As a last topic, we consider the behavior of maxmin auctions as the number of bidders grows large and the value distribution is held fixed. As the number of bidders tends to infinity, the optimal profit guarantee converges to the ex ante gains from trade. This generalizes the result of Du (2018) that shows the analogous result when there is common knowledge of non-negative gains from trade. We show that the optimal rate for this convergence is $O(1/\sqrt{N})$, and we show that the limit is attained even with efficient mechanisms. Finally, we show that the optimal profit guarantee converges to the ex ante gains from trade *even if the prior is misspecified*. Thus, whether the prior is correct is immaterial when the number of bidders is large.

The maxmin modeling approach allows us to identify new mechanisms with desirable theoretical properties, namely the sharp and unimprovable lower bound on profit, which holds uniformly across information structures and equilibria. A trade-off is the conceptual tension between the extreme ambiguity aversion of the Seller and the common knowledge among the agents with respect to the information structure. In particular, why does the Seller not simply ask the agents to report the information structure? In our view, the information structure of the agents is an as-if description of behavior, which we hope is a reasonable approximation. But we do not want to interpret it as something explicit that could be easily communicated by the agents to the Seller. In other words, we think the present modeling approach respects real-world limitations on what agents can articulate and communicate about their beliefs. The maxmin mechanism has a low-dimensional bidding interface and does not require the agents to report those beliefs, nor does it require the Seller to input a model of beliefs in order to compute the maxmin mechanism. That being said, the assumption of large ambiguity is as extreme as the assumption that the Seller knows the information structure exactly. We view it as a benchmark and a starting point for future work on informationally-robust optimal auctions. We return to this point in the conclusion of the paper.

1.3 Related literature

This paper lies at the intersection of the literatures on mechanism design and information design. We build on the seminal paper of Myerson (1981) on optimal auction design, and also subsequent work by Bulow and Klemperer (1996). Revenue equivalence arguments feature prominently in our analysis of optimal mechanisms on the minmax type space, and the revelation principle is a key idea behind the construction of the maxmin mechanism. We also draw heavily from the literature on robust predictions and specifically the incomplete information solution concept *Bayes correlated equilibrium* (BCE) (Bergemann and Morris, 2013, 2016). This is used in deriving the aforementioned differential equations that motivate the maxmin mechanism.

The most closely related papers are Du (2018) and Bergemann, Brooks, and Morris (2016). Du (2018) solves our maxmin auction design problem in the limit as the number of bidders goes to infinity and when the production cost is zero.[1] Specifically, Du constructs a sequence of mechanisms, one for each number of bidders, and associated lower bounds on profit that converge to the expected surplus as the number of bidders tends to infinity. The proof of the result uses duality arguments that are related to the ones we employ. While the mechanisms from Du (2018) are optimal in the many-bidder limit, they do not achieve the optimal profit guarantee when the number of bidders is finite and more than one. In contrast, Bergemann, Brooks, and Morris (2016) solves our maxmin auction design problem for the special case where there are exactly two bidders and two possible values, one for which the gains from trade are positive and one for which the gains are exactly zero. The proof strategy of Bergemann et al. shares some features with the one employed here, in that they construct a saddle point consisting of a maxmin mechanism and a minmax information structure, and they also use duality arguments to bound profit for the maxmin mechanism.[2] Our contribution is to provide a flexible and general theory of maxmin auctions and also to give a clearer understanding of the essential properties that characterize maxmin auctions, namely the double revelation principle and the role of the aggregate allocation sensitivity and the aggregate excess growth.

Chung and Ely (2007), Yamashita (2016), and Chen and Li (2018) also study maxmin auction design when the Seller does not know the information structure but when values are private and when the Seller preferred equilibrium is selected. In contrast, we focus on a common value environment. Other conceptually related studies of robust auction design are Neeman (2003), Brooks (2013), Yamashita (2015), Carroll (2016), and the literature on algorithmic mechanism design (e.g., Hartline and Roughgarden, 2009).

The rest of the paper proceeds as follows. Section 2 describes our model and solution concept. Section 3 presents an informal derivation of the strong maxmin solution. It is in this part of the paper that we describe the aforementioned system of differential and integral equations that our solution is constructed to satisfy. Section 4 presents our formal construction and characterization. Section 5 argues that all well-behaved strong maxmin solutions have the same profit guarantee, and describes a collection of saddle point problems that are solved by this solution. Section 6 discusses welfare in the many-bidder limit, and Section 7 concludes.

[1] Du (2018) also solves the present problem in the case of one bidder. With one bidder and binary values, our model reduces to that of Carrasco et al. (2018).

[2] Interestingly, the minmax information structure they identify coincides with the one we construct, but the maxmin mechanisms are different.

2 Model

2.1 Primitives

A unit of a good can be sold to one of N bidders. The bidders have a pure common value for the good v which is distributed according to the cumulative distribution function H on $\mathbb{R}_+$. Let V be the support of H. We assume that V is bounded, with $\underline{v}$ and $\overline{v}$ denoting the minimum and maximum, respectively. We also assume that $\underline{v} < \overline{v}$. If not, then the value is common knowledge, and the Seller can easily extract all the surplus.

The bidders have preferences over probabilities of receiving the good q_i and the amount they pay for it t_i, which are represented by the state-dependent utility index $vq_i - t_i$.

There is a constant cost of production $c \geq 0$. The Seller's profit from the profile of allocations $q = (q_1, \ldots, q_N)$ and transfers $t = (t_1, \ldots, t_N)$ is $\sum_{i=1}^{N}(t_i - cq_i)$. We make the further non-degeneracy assumption that the expected value is at least c, so that the ex ante expected gains from trade are non-negative.[3]

For technical reasons, we will assume that the left tail of H is not too thin. To state the precise condition, we will need the following definition: For a cumulative distribution F on $\mathbb{R}$, we define

$$F^{-1}(\alpha) = \min\{x \mid F(x) \geq \alpha\}$$

to be the *quantile function* for F. Because F is increasing and continuous from the right, there is a closed set of values that have a higher cumulative probability than α, so this minimum is well-defined. Moreover, F^{-1} is an increasing function and is continuous from the right, and it has discontinuities when there are gaps in the support of F.

Now, let G_N denote the distribution of the sum of N independent draws from the exponential distribution with unit arrival rate. (This object will feature prominently in the analysis.) We assume that there exists a $\varphi > 1$ such that

$$\limsup_{x \to 0} \frac{H^{-1}(G_N(x)) - \underline{v}}{x^{\varphi}} < \infty, \tag{1}$$

This condition is trivially satisfied when there is a mass point at $\underline{v}$, and it is also satisfied whenever H has a density that is bounded away from zero around $\underline{v}$.[4]

2.2 Information

Fix cumulative distributions F_1 and F_2. Recall that the distribution F_1 is a *mean-preserving spread* of F_2 if there exists if there exist a probability space and random

[3]Otherwise, a trivial solution to the maxmin problem is that the bidders have no information about the value and the Seller chooses to keep the good.

[4]If this is the case, then $H(v)$ must grow at least linearly in v around $\underline{v}$, so $H(v) \geq b(v - \underline{v})$ for some $b > 0$. As a result, $H^{-1}(G_N(x)) - \underline{v} \leq G_N(x)/b$. Given the explicit formulae for G_N and its density, g_N, in equations (13) and (14), it is clear that $G_N(x)/x^{\varphi} \to 0$ as long as $\varphi > 1$ and $N > 1$.

variables X_1 and X_2 such that X_1 has distribution F_1, X_2 has distribution F_2, and $\mathbb{E}[X_1|X_2] = X_2$. An equivalent characterization is that

$$\int_{y=-\infty}^{x} (F_1(y) - F_2(y))dy \geq 0, \tag{2}$$

for all $x \in \mathbb{R}$ and the left-hand side is exactly equal to zero when $x = \infty$ (Blackwell and Girshick, 1954; Rothschild and Stiglitz, 1970).

A *information structure* $\mathcal{S}$ consists of (i) a measurable set S_i of signals for each bidder i, (ii) a joint distribution $\pi \in \Delta(S)$ where $S = \times_{i=1}^{N} S_i$, and (iii) an interim value function $w : S \to \mathbb{R}$ such that H is a mean-preserving spread of the distribution of $w(s)$. For a profile of signals s, $w(s)$ is interpreted as the interim expectation of v conditional on s.[5]

2.3 Mechanisms

A *mechanism* $\mathcal{M}$ consists of measurable sets of messages M_i for each i and measurable mappings

$$q_i : M \to [0, 1], \quad t_i : M \to \mathbb{R}$$

for each i, where $M = \times_{i=1}^{N} M_i$ is the set of message profiles, such that

$$\sum_{i=1}^{N} q_i(m) \leq 1.$$

For technical reasons, we will assume that t_i is bounded below (although it may be negative). We further restrict attention to mechanisms that satisfy a condition we call *participation security*: For every i, there exists $0 \in M_i$ such that

$$vq_i(0, m_{-i}) - t_i(0, m_{-i}) \geq 0$$

for every $v \in V$ and every $m_{-i} \in M_{-i}$. By sending this message, bidder i ensures herself a non-negative payoff ex post, no matter what messages are sent by the other bidders.

2.4 Equilibrium

A mechanism $\mathcal{M}$ and an information structure $\mathcal{S}$ comprise a game of incomplete information. A *(behavioral) strategy* for bidder i is a transition kernel

$$\beta_i : S_i \to \Delta(M_i).$$

A profile of strategies $\beta = (\beta_1, \ldots, \beta_N)$ is identified with a transition kernel that associates to each $s \in S$ the product distribution $\beta_1(s_1) \times \cdots \times \beta_N(s_N)$ on $\Delta(M)$.

[5]This definition is equivalent to one in which we specify the joint distribution of the signals and the value. The interim expectation is the a object in our analysis, which is why we treat it as a primitive in our definition of an information structure.

Given a strategy profile β, bidder i's payoff is

$$U_i(\beta, \mathcal{M}, \mathcal{S}) = \int_S \int_M (w(s)q_i(m) - t_i(m))\beta(dm|s)\pi(ds).$$

Note that since w, q, and $-t$ are all bounded above, this integral is always well-defined. A strategy profile β is a *(Bayes Nash) equilibrium* if for all i and strategies β_i',

$$U_i(\beta, \mathcal{M}, \mathcal{S}) \geq U_i(\beta_i', \beta_{-i}, \mathcal{M}, \mathcal{S}).$$

The set of equilibria is denoted by $B(\mathcal{M}, \mathcal{S})$. Expected profit is

$$\Pi(\beta, \mathcal{M}, \mathcal{S}) = \int_S \int_M \sum_{i=1}^{N} (t_i(m) - cq_i(m))\beta(dm|s)\pi(ds).$$

2.5 Solution concept

Informally, we wish to to identify mechanisms that provide the best possible profit guarantee across all information structures and equilibria. We can think of as a zero-sum game between the Seller, who chooses the mechanism to maximize equilibrium profit, and Nature, who adversarially chooses the information structure to minimize equilibrium profit. But in order to rigorously define this game, we would have to specify which equilibrium is played, when there are multiple, and what the players' payoffs should be if no equilibrium exists.

Our analysis will lead to what we consider an exact solution to this joint mechanism design and information design problem. Equilibrium existence will not be an issue at the saddle point. More surprisingly, neither the Seller nor Nature can profitably deviate from the solution we construct, regardless of how we select the equilibrium that is played after a deviation. In that sense, the problem seems to favor an impartial perspective, where we are not prejudiced towards either the Seller or Nature in equilibrium selection. This spirit is captured by the following solution concept.

A *strong maxmin solution* of the joint mechanism design and information design problem consists of a triple $(\mathcal{M}, \mathcal{S}, \beta)$ of a mechanism, an information structure, and a strategy profile, with profit $\Pi = \Pi(\beta, \mathcal{M}, \mathcal{S})$, such that the following are satisfied:

1. For any information structure $\mathcal{S}'$ and any equilibrium β' of $(\mathcal{M}, \mathcal{S}')$, $\Pi \leq \Pi(\beta', \mathcal{M}, \mathcal{S}')$;

2. For any mechanism $\mathcal{M}'$ and any equilibrium β' of $(\mathcal{M}', \mathcal{S})$, $\Pi \geq \Pi(\beta', \mathcal{M}', \mathcal{S})$;

3. β is an equilibrium of $(\mathcal{M}, \mathcal{S})$.

We refer to Π as the *profit guarantee* of the solution.

Conditions 1 and 2 say that the Seller and Nature cannot improve their payoff by deviating, even if the deviator select the equilibrium. Condition 3 says that the guarantee is not vacuous, and there exists an equilibrium at which Π is attained. In fact, the definition implies that for a solution $(\mathcal{M}, \mathcal{S}, \beta)$, all equilibria of $(\mathcal{M}, \mathcal{S})$ must generate profit equal to Π.

Our main result is the construction of strong maxmin solution (Theorem 1). Note that the definition leaves open the possibility of solutions $(\mathcal{M}, \mathcal{S}, \beta)$ for which equilibria will fail to exist on $\mathcal{M}$ and $\mathcal{S}$ for a wide range of information structures and mechanisms, which effectively limits the set of deviations that are considered. The mechanism and information structure we construct will, however, be sufficiently well-behaved that an equilibrium exists for every alternative mechanism or information structure, as long as the set of messages or the set of signals, respectively, is finite. We call a solution with this property *regular*. Moreover, all regular solutions must have the same profit guarantee (Theorem 3). After the uniqueness result, we will present a collection of maxmin mechanism design and minmax information design that are solved by our strong maxmin solution (Corollaries 1 and 2).

3 A roadmap to the solution

We will give a complete construction a strong maxmin solution at the beginning of Section 4. Theorem 1 will then verify that the construction is indeed a solution. The construction and proof are somewhat intricate. This section gives an informal derivation and explanation of our solution. To be clear, our purpose is to develop intuition, and the proof of Theorem 1 in no way depends on the following discussion. In fact, many of the concepts we now introduced are used only implicitly in the formal arguments.

3.1 The structure of the solution

The strong maxmin solution we construct will be denoted $(\overline{\mathcal{M}}, \overline{\mathcal{S}}, \overline{\beta})$. The high level structure is as follows. The signals for the information structure and the messages for the mechanism are elements of $\overline{M}_i = \overline{S}_i = [0, \infty]$, the extended real line. Thus, a common language is used for signals and messages. In addition, the equilibrium strategies specify that each bidder send a message that is equal to their signal: for all i and s_i,

$$\overline{\beta}_i(s_i) = s_i.$$

Thus, one interpretation of the solution is that the maxmin mechanism $\overline{\mathcal{M}}$ is a *direct mechanism* on the minmax information structure $\overline{\mathcal{S}}$, in which messages are normalized to be equal to signals and bidders report truthfully in equilibrium. An equivalent interpretation is that $\overline{\mathcal{S}}$ is a *Bayes correlated equilibrium* (BCE) on $\overline{\mathcal{M}}$, in which the signals are "recommendations" of a message to send, and in equilibrium, agents follow their recommendation.

If we held the information structure fixed and maximized profit across mechanisms and equilibria, then the well-known revelation principle (Myerson, 1981) says that it is without loss of generality to restrict attention to direct mechanisms. Similarly, if the mechanism were fixed and we minimized profit across information structures and equilibria, then it is without loss of generality to restrict attention to BCE (Bergemann and Morris, 2013, 2016), which is a kind of revelation principle for games. In the present model, both the mechanism and the information structure are endogenous objects, so the standard revelation arguments do not apply.[6] It is therefore a surprising *result* that there exists a solution that admits the same normalization. We refer to this as the *double revelation principle.*

3.2 The minmax information structure

We next describe the rest of the minmax information structure $\overline{\mathcal{S}}$, from which we will subsequently derive the maxmin mechanism. The form of $\overline{\mathcal{S}}$ can be understood using the celebrated *revenue-equivalence formula* of Myerson (1981), suitably adapted to the common value setting.

Consider an information structure with independent real signals s_i with corresponding densities f_i. Revenue equivalence says that expected profit from a direct mechanism is, up to a constant, the expectation of the *virtual value* of the bidder who receives the good. When the value function is differentiable, the virtual value of bidder i when the signal profile is s is given by[7,8]

$$\psi_i(s) = w(s) - c - \frac{1 - F_i(s_i)}{f_i(s_i)} \frac{\partial w(s)}{\partial s_i},$$

and where F_i cumulative distribution of bidder i's signal. Thus, the virtual value is equal to the gains from trade, minus the inverse hazard rate times the sensitivity of the value to bidder i's signal. The inverse hazard rate is the measure of higher types who receive an information rent from being able to mimic s_i, relative to the likelihood of s_i, while $\partial w(s)/\partial s_i$ quantifies the value of bidder i's private information at the profile s.

Among such information structures, it is without loss of generality to normalize the signals to be exponential with a unitary arrival rate:

$$F_i(x) = 1 - \exp(-x).$$

As a result, the inverse hazard rate is constant and equal to one, and drops out of the virtual value formula. The remaining degree of freedom in specifying information is the value function $w(s)$.

We may ask, among such information structures, which one would be the worst case for the Seller? Drawing on intuition from zero-sum games, we might suspect that the

[6] In particular, for a fixed information structure, any mechanism $\mathcal{M}$ and equilibrium β has a corresponding direct mechanism $\mathcal{M}'$ in which truth telling is an equilibrium. But the mechanism $\mathcal{M}'$ may have other equilibria with no counterpart in $\mathcal{M}$, and our solution concept imposes strong conditions on how profit varies across *all* equilibria. Similarly, replacing a given information structure and equilibrium with the corresponding direct information may lead to a different set of equilibria.

[7] In the classic formulation of Myerson (1981), bidder i's virtual value is their value minus the inverse hazard rate. We would obtain this formula result if there were bidder-specific values $w_i(s)$ and we normalizes signals so that $w_i(s) = s_i$, in which case the partial derivative is identically one. The formula reported here is a special case of one that appears in Bulow and Klemperer (1996).

[8] Our formal arguments in Section 4 sidestep the direct calculation of virtual values, in order to avoid technical complications associated with whether the integral representation of the bidders' indirect utilities holds.

worst case would be associated with indifference between lots of mechanisms. This would roughly mean that $\overline{\mathcal{S}}$ is hard to best respond to, in the sense that while lots of mechanisms perform reasonably well, no mechanism stands out as exceptional.

In fact, it is easy to engineer the value function so that the virtual value is the same for all bidders, and hence conditional on allocating the good, the Seller is indifferent as to who it is allocated to. This happens when the interim value is of the form[9]

$$w(s) = w(\Sigma s),$$

where Σs is the *aggregate signal*:

$$\Sigma s = s_1 + \cdots + s_N.$$

(We will maintain this notational convention for the sum of a vector's elements throughout the paper.) As a result, the interim expected value is equally sensitive to all signals, and all bidders have the same virtual value of $w(\Sigma s) - c - w'(\Sigma s)$.

Among such information structures, we are still free to choose the particular function of the aggregate signal. An important variant of our model, which we discuss in Section 4.3, is the *must-sell case*, where the good has to be sold with probability one. This is in contrast to the *can-keep case*, in which the Seller can withhold the good. Note that the aggregate signal has the cumulative distribution G_N mentioned previously in the statement of the left-tail condition in Section 2, and we let g_N denote the associated density.[10] Since all bidders have the same virtual value, profit in the must-sell case is

$$\int_{x=0}^{\infty} (w(x) - c - w'(x) g_N(x) dx. \tag{3}$$

This formula assumes that transfers are set so that participation security binds. Note that the expectation of $w(x)$ is pinned down from H and would be the same with any feasible value function. Thus, to minimize profit, we should pick the value function that has the largest expected slope. The expected slope is maximized by the *fully-revealing value function*:

$$\widehat{w}(x) = H^{-1}(G_N(x)),$$

where H^{-1} is the quantile function corresponding to H. In words, $\widehat{w}$ matches aggregate signals and values so that the percentile of the aggregate signal is always equal to the percentile of the value. This value function is fully revealing in the sense that there is no uncertainty about the value, conditional on the join of the bidders' information. It is intuitive that the fully-revealing value function minimizes profit, since it maximizes the amount of private information the bidders have about the value. Figure 1 illustrates the fully-revealing value and virtual value functions when $c = 0$ and v is standard uniform, $\widehat{w}(x)$ is equal to $G_N(x)$, which consist of a gray segment below x^* and a black segment above x^*.

[9]We hope we will not create confusion by using the same notation for the interim value as a function of the signal profile and for the interim value as a function of the aggregate signals.

[10]Recall that the signals are independent draws from the standard exponential distribution. Both G_N and g_N have closed-form expressions, given as equations (13) and (14) below.

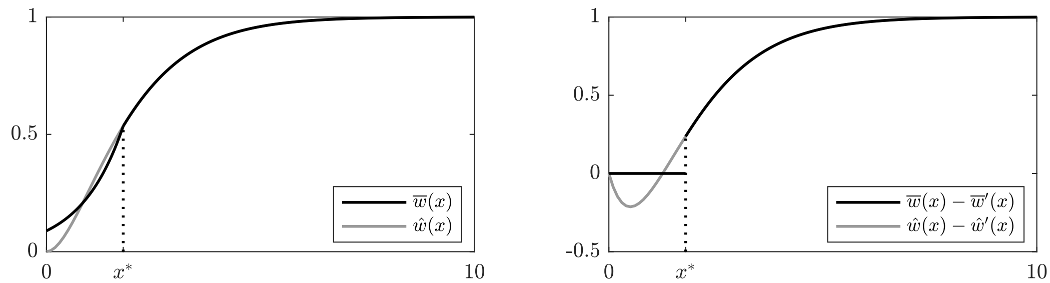

Figure 1: $N = 2$, $v \sim U[0, 1]$, and $c = 0$. The minmax value functions in the must-sell and can-keep cases coincide above x^*. The fully-revealing value function (in blue) is equal to G_2.

For some distributions, $\widehat{w}$ will also be the minmax value function, even if the good does not have to be sold. This is *not* the case for the uniform distribution. Examining the right-hand panel of Figure 1, we can see that the virtual value is strictly negative when the aggregate signal is low, so that the Seller would strictly prefer to withhold the good at such profiles. The Seller can be made strictly worse off by adding some noise to the bidders' information to create additional indifference on the part of the Seller, between selling and not selling. For this to be the case, the virtual value must be exactly zero:

$$w(x) - c - w'(x) = 0,$$

i.e., the interim expected gains from trade, denoted $\gamma(x) = w(x) - c$, is of the form $k \exp(x)$ for some positive constant k. The function $\gamma(x)$, which we refer to as the *gains function*, will turn out to be a key object in our analysis.

The exponential shape for the gains function can be achieved by adding noise to the signal, so that we effectively pool realizations where the virtual value has different signs. In the uniform example, we can replace the fully-revealing gains function $\widehat{\gamma}(x) = \widehat{w}(x) - c$ when the value is low with an exponential shaped segment on an interval $[0, x^*]$. Both $\widehat{\gamma}(0)$ and x^* are chosen so that H remains a mean-preserving spread of the distribution of the interim expected value, and so that the exponential shape connects continuously with the fully-revealing gains function. We denote this new gains function by $\overline{\gamma}$, and its associated value function is denoted $\overline{w}$. In fact, the $\overline{w}$ depicted in black in Figure 1 is the minmax gains function when the Seller can keep the good and when the cost is zero.

More generally, the sign of the fully-revealing virtual value might switch back and forth. In Section 4.1, we describe a general procedure that transforms the fully-revealing gains function so that the resulting virtual value is everywhere non-negative, and it is always weakly optimal to allocate the good. We refer to this as *grading the gains function*, meaning we decrease the derivative of the gains function so that it does not grow faster than exponential. The graded gains and value functions are denoted by $\overline{\gamma}$ and $\overline{w}$, respectively, and the resulting information structure is denoted $\overline{\mathcal{S}}$. Proposition 1 characterizes profit for this construction and gives a generalized upper bound $\overline{\Pi}$ for the optimal profit guarantee, which is simply

$$\overline{\Pi} = \int_{x=0}^{\infty} \overline{\gamma}(x) g_{N-1}(x) dx. \tag{4}$$

This formula can be obtained from (3) via integration by parts, if we replace $w(x) - c$ with $\overline{\gamma}$ and use the identity that $dg_N(x)/dx = g_{N-1}(x) - g_N(x)$. We can alternatively interpret $\overline{\Pi}$ as the highest posted price at which all bidders would be willing to purchase the good, regardless of their signals. Such a posted price would always allocate the good and make participation security bind, and hence it is an optimal mechanism on $\overline{\mathcal{S}}$ (although it is not a maxmin mechanism!)

3.3 Sufficient conditions for an optimal profit guarantee

We will presently derive a maxmin mechanism $\overline{\mathcal{M}}$ from the minmax information structure $\overline{\mathcal{S}}$. At first glance, $\overline{\mathcal{S}}$ does not seem to tell us very much about $\overline{\mathcal{M}}$, because so many mechanisms are optimal. We will learn quite a bit, however, by focusing our attention on direct mechanisms on $\overline{\mathcal{S}}$ for which $\overline{\mathcal{S}}$ induces a profit minimizing BCE and for which $\overline{\Pi}$ is minimum profit.

Let us define an *outcome* of such a mechanism to be a joint distribution over values and message profiles $\sigma \in \Delta(V \times \overline{M})$. Every information structure and equilibrium of this mechanism can be associated with an outcome σ, where likelihood of v and m is simply the expectation, across s, of the likelihood that m is the message profile induced by the bidders' realized signals and strategies and such that v is the value drawn from the a conditional distribution of v given $w(s)$ (which exists by the mean-preserving spread condition). The set of outcomes that can be induced by some information structure and equilibrium is equivalent to the set of *Bayes correlated equilibria* (BCE). A BCE is an outcome that satisfies: *obedience constraints*, which say that for all i and m_i, m_i is a best response to the conditional distribution of (v, m_{-i}) induced by σ and conditioning on m_i; and *marginal constraints*, that the marginal distribution on V is the prior H.

Let us denote by $\overline{\sigma}$ an outcome induced by $(\overline{\mathcal{S}}, \overline{\beta})$. We are considering mechanisms such that $\overline{\sigma}$ is the profit-minimizing BCE of the mechanism and that profit in this BCE is $\overline{\Pi}$. It will turn out that the only obedience constraints that are relevant for our problem are those associated with local optimality, i.e., that for all i and m_i,

$$\int_{V \times \overline{M}_{-i}} \left(v \frac{\partial q_i(m_i, m_{-i})}{\partial m_i} - \frac{\partial t_i(m_i, m_{-i})}{\partial m_i} \right) \sigma(dv, dm_{-i} | m_i) = 0.$$

The BCE $\overline{\sigma}$ is therefore the solution to an infinite dimensional linear program, for which the associated Lagrangian is[11]

$$\begin{aligned}\mathcal{L}(\sigma, \{\alpha_i\}, \lambda) = &\sum_{i=1}^{N} \int_{V\times M} (t_i(m) - cq_i(m))\, \sigma(dv, dm) \\ &+ \sum_{i=1}^{N} \int_{V\times M} \alpha_i(m_i) \left(v\frac{\partial q_i(m)}{\partial m_i} - \frac{\partial t_i(m)}{\partial m_i} \right) \sigma(dv, dm) \\ &+ \int_{V\times M} \lambda(v) \left(H(dv) - \sigma(dv, dm) \right).\end{aligned} \tag{5}$$

This Lagrangian has three terms: profit induced by the BCE, the sum of local obedience constraints times their corresponding multipliers (the functions α_i), and the sum of marginal constraints times their corresponding multipliers (the function λ). A necessary first-order condition for $\overline{\sigma}$ to be the profit-minimizing BCE is that for all (v, m),

$$\sum_{i=1}^{N} \left[t_i(m) - cq_i(m) + \alpha_i(m_i) \left(v\frac{\partial q_i(m)}{\partial m_i} - \frac{\partial t_i(m)}{\partial m_i} \right) \right] - \lambda(v) \geq 0, \tag{6}$$

with the constraint holding as an equality for (v, m) in the support of $\overline{\sigma}$.

It turns out that the structure of $\overline{\mathcal{S}}$ implies particular values for the multipliers α_i and λ. Based on the envelope theorem, we can guess that $\lambda(v)$ is the derivative of minimum profit in the maxmin mechanism with respect to the prior probability of v. This should coincide with the derivative of $\overline{\Pi}$ with respect to the probability of v, which we denote by $\overline{\lambda}(v)$. If not, then by making v either more or less likely, we could make minimum profit from the maxmin mechanism increase faster than $\overline{\Pi}$. The function $\overline{\lambda}$ has an explicit formula given in equation (19) below, and we will shortly use the fact that $\overline{\lambda}$ is concave.[12]

As for the multipliers on local obedience, there is an even simpler answer: $\alpha_i(m_i)$ is constant and equal to 1 for all i. The reason is that the Lagrangian (5) is very similar to the Lagrangian for the linear program of *maximizing* profit given the *fixed* information structure $\overline{\mathcal{S}}$, where we hold fixed $\sigma = \overline{\sigma}$ and optimize over (q_i, t_i), and local obedience is reinterpreted as local incentive compatibility. As is well known, local incentive constraints bind at the solution, and the optimal multiplier on local incentive compatibility is equal to the inverse hazard rate of the signal. For $\overline{\mathcal{S}}$, this hazard rate is constant and equal to 1 due to the standard exponential normalization.

By substituting in these multipliers, equation (6) reduces to a constraint on the maxmin allocation and transfer rules. If we denote the aggregate supply by $Q(m) = \Sigma q(m)$, then (6) is equivalent to for all (v, m),

$$\nabla \cdot t(m) - \Sigma t(m) \leq v\nabla \cdot q(m) - \overline{\lambda}(v) - cQ(m), \tag{7}$$

[11]Since we are just providing informal motivation, we will not rigorously argue that this Lagrangian is equivalent to the profit-minimization program. The fact that this is the correct Lagrangian is a consequence of Theorem 1 below.

[12]The optimal value multiplier must be concave, because if, for every v, (6) holds as an equality for some m, then it must be that $\lambda(v)$ is the minimum of a collection of linear function of v, indexed by m.

where $\nabla\cdot$ is the divergence operator. Recall that this constraint must bind on the support of $\overline{\sigma}$, which, in particular, includes pairs (v, m) such that $v = \widehat{w}(\Sigma m)$. For a fixed m, this value must minimize the right-hand side of (7), and since $\overline{\lambda}$ is concave, it must be that

$$\nabla \cdot q(m) = \overline{\lambda}'(\widehat{w}(\Sigma m)). \tag{8}$$

We refer to the left-hand side of (8) as the *aggregate allocation sensitivity.* In fact, $\overline{\lambda}'(\widehat{w}(x))$ can be computed in closed form, and we denote it by $\overline{\mu}(x)$. When the value function is full-revealing, $\overline{\mu}(x) = (N-1)/x$, and on an interval where the value function is graded, $\overline{\mu}$ is a constant that just depends on the end points of the interval. The exact formula is given in equation (15) below.

Once we substitute in the optimal aggregate allocation sensitivity, equation (7) reduces to a condition on the transfers:

$$\nabla \cdot t(m) - \Sigma t(m) = \widehat{w}(\Sigma m)\overline{\mu}(\Sigma m) - \overline{\lambda}(\widehat{w}(\Sigma m)) - cQ(m). \tag{9}$$

The left-hand side of (9) the *aggregate excess growth*, i.e., difference between how fast the bidders' transfers grow in their own messages relative to exponential growth. We refer to equation (9) as *profit-incentive alignment* (PIA), since it imposes a linkage between ex post profit, $\Sigma(t-cq)$, and the sum of the bidders' local incentive constraints, $v\nabla \cdot q - \nabla \cdot t$. This ensures that as long as local incentive constraints are satisfied, profit cannot fall too low.

We have been using the profit-minimization program to derive necessary conditions on a maxmin mechanism. But as we argue in Proposition 2, these conditions are actually *sufficient* for a mechanism to guarantee profit of at least $\overline{\Pi}$. Specifically, if we have a mechanism with differentiable allocation and transfer rules and such that that (i) the aggregate allocation sensitivity is $\overline{\mu}$ and (ii) the aggregate excess growth and the aggregate supply satisfy (9), then profit must be at least $\overline{\Pi}$ in all information structures and all equilibria. The proof is essentially an application of the weak duality.

3.4 Construction of a maxmin mechanism

The last step is to explicitly construct allocation and transfer rules that satisfy (8) and (9) and such that truth telling is an equilibrium at $\overline{\mathcal{S}}$. Note that the Proposition 2 does not imply that truth telling is an equilibrium, and this is something we will have to verify for the mechanism we construct.

An allocation rule with the desired aggregate allocation sensitivity is relatively easy to guess. Consider the case with two bidders and zero cost. If we assume that $Q(m) = 1$, then $q_2(m_1, m_2) = 1 - q_1(m_1, m_2)$, and aggregate allocation sensitivity reduces to

$$\frac{\partial q_1(m_1, m_2)}{\partial m_1} - \frac{\partial q_1(m_1, m_2)}{\partial m_2} = \frac{1}{m_1 + m_2}.$$

Now consider a level curve where $m_1 + m_2 = x$. Then we can view the left-hand side as the total derivative with respect to m_1 along the parametric curve $m_2(m_1) = x - m_1$, so that integrating both sides, we obtain

$$q_1(m_1, m_2) = \frac{m_1}{x} + C(x).$$

In order to have $q_1 \in [0,1]$, we must have $C(x) = 0$, so the allocation probability is simply the bidder's share of the aggregate message.

More generally, equation (8) is satisfied by the following *proportional allocation rule*:

$$q_i(m) = \begin{cases} \frac{m_i}{\Sigma m} Q(\Sigma m) & \Sigma m > 0, \\ \frac{1}{N} Q(0) & \Sigma m = 0, \end{cases}$$

where Q is a continuous and almost everywhere differentiable aggregate supply (which only depends on the aggregate message). This allocation rule has a aggregate allocation sensitivity that only depends on Q. In equation (15), we give an explicit formula for an aggregate supply function $\overline{Q}$ such that the induced aggregate allocation sensitivity is $\overline{\mu}$, which is 1 whenever the value function is fully revealing, as it must be to maximize profit on $\overline{\mathcal{S}}$. This $\overline{Q}$ then defines the maxmin allocation rule, which we denote by $\overline{q}$.

This leaves the transfers. Since $\overline{Q}$ has been specified, we can denote by $\overline{\Xi}$ the target aggregate excess growth, which is equal to the right-hand side of (9) and is just a function of the aggregate message. In Section 4, we will present a general solution to this equation. The formula can be motivated as follows. Any solution to (9) must be associated with an apportionment of $\overline{\Xi}$ among the bidders, where

$$\xi_i(m) = \frac{\partial t_i(m)}{\partial m_i} - t_i(m) \tag{10}$$

is bidder i's share of the excess growth. Assuming for now that $t_i(0, m_{-i}) = 0$,[13] we can integrate (10) to obtain the representation

$$t_i(m) = \exp(m_i) \int_{x=0}^{m_i} \exp(-x) \xi_i(x, m_{-i}) dx. \tag{11}$$

Thus, the problem is to choose ξ_i so that $\Sigma \xi_i(m) = \overline{\Xi}(m)$.

At first glance, there seems to be tremendous flexibility in how we divide the aggregate excess growth. The danger lurking here is that there is no guarantee, for an arbitrarily choice of ξ_i to satisfy (9), that an equilibrium will exist on *any* information structure, let alone $\overline{\mathcal{S}}$. As a result, the profit lower bound implicit in (9) may be vacuous. At a high level, this is related to well-known issues of equilibrium non-existence in games with discontinuous payoffs and/or non-compact action spaces. This suggests that we should impose a kind of transversality condition on the transfers, so that they are continuous

[13] Note that with the proportional allocation rule, $q_i(0, m_{-i}) = 0$ as long as $\Sigma m_{-i} > 0$. Thus, aside from the zero message profile which is exceptional, participation security reduces to $t_i(0, m_{-i}) \leq 0$. For the solution we define below, $t_i(0) = \underline{v} q_i(0)$, which will be strictly positive when $\underline{v} > 0$.

and bounded at infinity. Inspection of equation (11) indicates that a necessary condition for the transfers to be bounded is that

$$\int_{m_i=0}^{\infty} \xi_i(m_i, m_{-i}) \exp(-m_i) dm_i = 0. \tag{12}$$

In other words, it must be that each bidder has zero excess growth in expectation across their own message, holding fixed the other bidders' messages. If the excess growth functions are themselves continuous and bounded, the transfers will be continuous and bounded on the extended real line, and we could apply standard equilibrium existence arguments. This property is used in Section 5 to argue that the profit guarantee of a strong maxmin solution unique among sufficiently well-behaved solutions.

Even though we motivated (12) in terms of abstract conditions for equilibrium existence, it turns out to be closely related to incentive compatibility on $\overline{\mathcal{S}}$. The usual Myersonian revenue equivalence arguments, applied to $\overline{S}$, tell us that that if the allocation $\overline{q}$ is implemented, then the implementing interim transfers are completely pinned down from local incentive compatibility. When there are two bidders, equation (12) is *equivalent* to interim incentive compatibility of $\overline{q}$, and more generally, it is a sufficient condition. We discuss this further in the proof of Proposition 3 and footnote 16.

The remaining question is whether there *exists* a transfer function that satisfies equations (9) and (12). At first glance, these equations seem to contradict one another: If the bidders split $\overline{\Xi}$ between themselves as per (9), how can every bidder get zero excess growth on average as per (12)? The way out of the "paradox" is that the ex ante expectation of $\overline{\Xi}$ turns out to be zero. Indeed, when there are two bidders, the following excess growth functions work:

$$\overline{\xi}_i(m) = \frac{1}{2}\left[\overline{\Xi}(m) - \int_{x=0}^{\infty} \overline{\Xi}(x + m_j) \exp(-x) dx + \int_{x=0}^{\infty} \overline{\Xi}(m_i + x) \exp(-x) dx\right].$$

We can interpret $\overline{\xi}$ as follows: Each bidder is allocated half of the excess growth, which is the first term inside the brackets. Without further modification, this would generally violate (12). The second term in the brackets "cancels out" bidder i's share of the aggregate excess growth on average across m_i so that (12) is satisfied. The final term in the brackets cancels out the counterpart of the middle term in $\overline{\xi}_j$, so that the aggregate excess growth is preserved. Finally, when we take an expectation across m_i, the last term reduces to the ex ante expectation of $\overline{\Xi}$, which is zero, so that it does not change the expected excess growth across m_i. These excess growth functions define transfer rules that satisfy (9) and (12), so that $\overline{\beta}$ is an equilibrium at $\overline{\mathcal{S}}$.

When there are more than two bidders, there is a generalization of this formula, which can be recovered by computing the excess growth for the transfer defined in (17) below. We should emphasize, however, that while the individual excess growths and the representation (11) are useful motivation for the transfer rule we construct, we actually give an explicit formula for the transfers in Section 4, and our subsequent arguments use the individual excess growths implicitly. Similarly, we do not make explicit use of BCE in the proof of Theorem 1. Nonetheless, these ideas are all at work "under the hood."

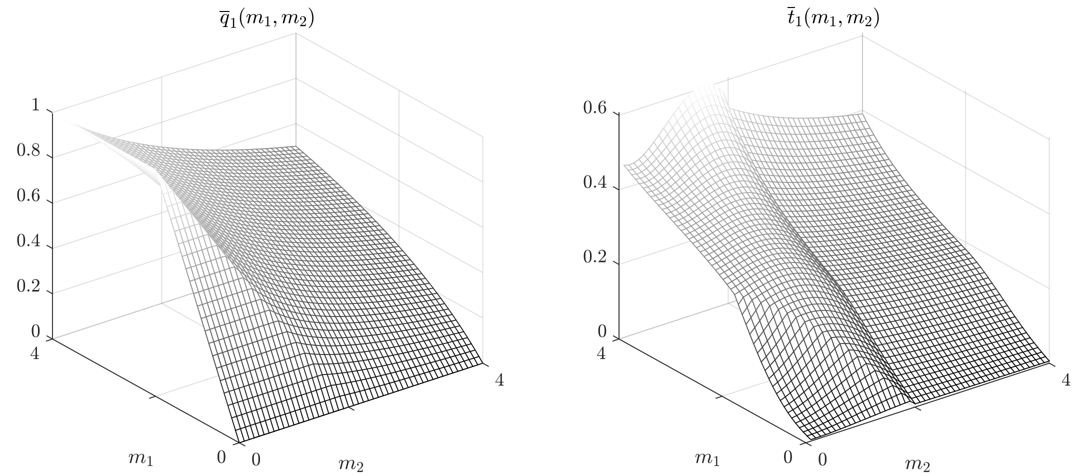

Figure 2: The maxmin mechanism and transfer rule in the uniform case with $N = 2$.

A non-trivial technical complication, which we have hitherto glossed over, is that the functions $\overline{\lambda}$ and $\overline{\Xi}$ may not be bounded as the aggregate message goes to zero. Indeed, (9) includes a term $\widehat{w}(x)\overline{\mu}(x)$, which necessarily blows up for x small when $\underline{v} > 0$ and the value function is not graded at 0. This is dealt with by breaking the transfer up into two pieces, a *base payment* $\overline{t}_i^b(m) = \underline{v}\overline{q}_i(m)$ and a *premium* $\overline{t}_i^p$. The excess growth from the base payment is $\underline{v}(\overline{\mu}(\Sigma m) - \overline{Q}(\Sigma m))$, and can be substituted into (9) to yield a *premium aggregate excess growth*, given by equation (18) below. The preceding discussion then applies to the construction of the premium transfers.

Theorem 1 shows that this construction completes the specification of a maxmin mechanism, and therefore completes the construction of a strong maxmin solution. The optimal allocation and transfer rules are plotted for the two-bidder/uniform/zero-cost example in Figure 2 for message profiles in $[0, 5]^2$.

4 A Strong Maxmin Solution

We now formally construct and characterize a strong maxmin solution to the joint mechanism design and information design problem. The structure of the arguments is quite different from the informal overview in Section 3. We will first completely construct the solution in Section 4.1. We then present our main theorem in Section 4.2, which asserts that constructed triple is indeed a strong maxmin solution. The proof immediately follows. Sections 4.3 and 4.4 discuss two special cases, when the good must be sold and when the value distribution is single crossing, respectively.

4.1 Construction of the solution

4.1.1 Minmax information

The minmax information structure $\overline{\mathcal{S}}$ is defined as follows. The N bidders have signal spaces $\overline{S}_i = [0, \infty]$, with the standard measurable structure. The distribution of the signals is

$$\overline{\pi}(ds) = \exp(-\Sigma s)ds.$$

In other words, the signals are independent draws from the exponential distribution with arrival rate 1.

The aggregate signal $x = \Sigma s$ has an Erlang distribution (which is a special case of the Gamma distribution) and has a probability density function

$$g_N(x) = \frac{x^{N-1}}{(N-1)!}\exp(-x) \tag{13}$$

and cumulative distribution function

$$G_N(x) = 1 - \sum_{n=1}^{N} g_n(x). \tag{14}$$

The interim expected value is a function of the aggregate signal is defined according to the following *grading procedure*. Recall that $\widehat{w}(x) = H^{-1}(G_N(x)$ is the full-revealing value function, and $\widehat{\gamma}(x) = \widehat{w}(x) - c$ is the fully-revealing gains function. Let

$$\widehat{\Gamma}(x) = \int_{y=0}^{x} \widehat{\gamma}(y) g_N(y) dy.$$

Also let

$$E(x) = \int_{y=0}^{x} \exp(y) g_N(y) dy,$$

which is strictly increasing, and hence it has a well-defined inverse E^{-1}. Let $\text{cav}(\widehat{\Gamma} \circ E^{-1})$ denote the smallest concave function that is everywhere above $\widehat{\Gamma} \circ E^{-1}$. We then set $\overline{\Gamma} = \text{cav}(\widehat{\Gamma} \circ E^{-1}) \circ E$, and define

$$\overline{\gamma}(x) = \frac{1}{g_N(x)} \frac{d}{dx} \overline{\Gamma}(x);$$
$$\overline{w}(x) = \overline{\gamma}(x) + c,$$

where the derivative is taken from the right. We refer to $\overline{\gamma}$ and $\overline{w}$ as the *graded gains function* and the *graded value function*, respectively.[14]

[14] Grading is evocative of "ironing" in Myerson (1981) and concavification in Kamenica and Gentzkow (2011). Grading is used to construct the bidders' information that minimizes the Seller's profit, subject to the Seller being always willing to allocate the good and subject to a mean-preserving spread constraint. In Myerson, ironing is used to construct the mechanism that maximizes the Seller's profit, subject to global incentive compatibility. In Kamenica and Gentzkow, concavification is used to construct a receiver's information to maximize a sender's payoff, subject to a mean-preserving spread constraint. We can find no tight link between these problems, beyond the very high-level connection of optimization subject to monotonicity and/or mean-preserving spread constraints.

4.1.2 Maxmin mechanism

We next construct the maxmin mechanism $\overline{\mathcal{M}}$. The message space is $\overline{M}_i = [0,\infty]$, the extended real line. Note that the infinite messages will not feature prominently in the analysis of this section, but it will be used to argue that the solution is regular in Section 5.

Let us define a *graded interval* to be an interval $[a,b]$ such that $\overline{\Gamma}(x) = \widehat{\Gamma}(x)$ for $x \in \{a,b\}$ and $\overline{\Gamma}(x) > \widehat{\Gamma}(x)$ for $x \in (a,b)$. The allocation rule is

$$\overline{q}_i(m) = \begin{cases} \frac{\overline{Q}(0)}{N} & \text{if } \Sigma m = 0; \\ \frac{m_i}{\Sigma m}\overline{Q}(\Sigma m) & \text{if } 0 < \Sigma m < \infty; \\ \frac{1}{|\{j|m_j=\infty\}|} & \text{if } \Sigma m = \infty, \end{cases}$$

where the aggregate supply function is given by

$$\overline{Q}(x) = \begin{cases} C(a,b)\frac{x}{N} + D(a,b)\frac{1}{x^{N-1}} & \text{if } x \in [a,b] \text{, where } [a,b] \text{ is a graded interval;} \\ 1 & \text{otherwise,} \end{cases} \tag{15}$$

and

$$C(a,b) = N\frac{b^{N-1} - a^{N-1}}{b^N - a^N};$$
$$D(a,b) = \frac{b-a}{b^N - a^N}a^{N-1}b^{N-1}.$$

We let $\overline{\mu}(\Sigma m)$ denote the aggregate allocation sensitivity for $\overline{q}$:

$$\overline{\mu}(x) = \begin{cases} C(a,b) & \text{if } x \in [a,b), \text{where } [a,b] \text{ is a graded interval,} \\ \frac{N-1}{x} & \text{otherwise.} \end{cases} \tag{16}$$

The transfer rule is decomposed into a base and a premium:

$$\overline{t}_i(m) = \underline{v}\,\overline{q}_i(m) + \overline{t}_i^p(m).$$

The premium $\overline{t}_i^p$ is defined as follows. Let Z be the set of all permutations of $\{1,\ldots,N\}$, and for a message profile $m \in \overline{M}$ and $\zeta \in Z$, we let $m_{\zeta \le k}$ denote the subvector of messages $m_{\{j|\zeta(j)\le k\}}$. Then

$$\overline{t}_i^p(m) = \frac{1}{N!}\sum_{\zeta\in Z}\int_{x=0}^{\infty}\left(\overline{\Xi}^p(\Sigma m_{\zeta<\zeta(i)} + x) - \overline{\Xi}^p(\Sigma m_{\zeta\le\zeta(i)} + x)\right) g_{N-\zeta(i)+1}(x))\,dx, \tag{17}$$

where

$$\overline{\Xi}^p(x) = \overline{\mu}(x)(\widehat{w}(x) - \underline{v}) - \overline{\lambda}\left(\widehat{w}(x)\right) + \left(\underline{v} - c\right)\overline{Q}(x), \tag{18}$$

is the premium aggregate excess growth and

$$\overline{\lambda}(v) = \overline{\Pi} + \int_{x=0}^{\infty}\overline{\mu}(x)G_N(x)d\widehat{w}(x) - \int_{\nu=v}^{\overline{v}}\overline{\mu}(G_N^{-1}(H(\nu)))d\nu, \tag{19}$$

and $\overline{\Pi}$ is given by (4).

4.1.3 Strategies

Finally, let $\overline{\beta}_i$ be the truthful strategy in the mechanism $\overline{\mathcal{M}}$ under the information structure $\overline{\mathcal{S}}$: For all i, $\overline{\beta}_i(s_i) = m_i$ for every $s_i = m_i \in \overline{S}_i = \overline{M}_i$.

This completes the construction of the solution..

4.1.4 Illustration

The objects used in the construction of the solution are illustrated in Figure 3. The particular value distribution used for this example is uniform on $[0, \widehat{v}] \cup [\widehat{v} + d, 1 + d]$, where $\widehat{v} \in (0, 1)$, and where there is a strictly positive cost of production.

The top row of figures illustrates the construction of the gains function. From left to right are the gains functions, integrated gains functions, and rescaled gains functions. The fully-revealing versions are in light-gray, and the graded versions are in black. The gap in the support of v corresponds to the discontinuity in the fully-revealing gains function at $\widehat{x} = G_N^{-1}(\widehat{v})$. We can see that the graded integrated gains function is obtained via a concavification that is depicted in the top right panel. There are two graded intervals, which are $[0, x_1]$ and $[x_2, x_3]$.

In the bottom row are pictures of various objects used in constructing $\overline{\mathcal{M}}$. Again, fully-revealing objects are in gray and graded counterparts are in black (see Section 4.3 for the discussion of the fully-revealing objects). We can see that the aggregate supply is 1 whenever the gains function is not graded, and it has a simple linear form on the lowest graded interval. This is always the case when there is a graded interval at 0. The aggregate allocation sensitivity is second from left. It is flat on graded intervals and equal to $(N - 1)/x$ everywhere else. The two right panels depict the value multiplier $\overline{\lambda}$ and the aggregate excess growth of the premium $\overline{\Xi}^p$.

4.2 Main result

The main result of the paper is the following:

Theorem 1 (Existence)**.** *The triple $(\overline{\mathcal{M}}, \overline{\mathcal{S}}, \overline{\beta})$ is a strong maxmin solution with a profit guarantee of $\overline{\Pi}$ defined by* (4)*.*

The theorem will follow from Propositions 1–3. Proposition 1 verifies that $\overline{\mathcal{S}}$ is a well-defined information structure and that no equilibrium and mechanism can generate more than $\overline{\Pi}$ in profit when the information structure is $\overline{\mathcal{S}}$. Proposition 2 verifies that $\overline{\mathcal{M}}$ is a well-defined mechanism and that no information structure and equilibrium can generate less than $\overline{\Pi}$ in profit when the mechanism is $\overline{\mathcal{M}}$. Finally, Proposition 3 verifies that $\overline{\beta}$ is an equilibrium of $(\overline{\mathcal{M}}, \overline{\mathcal{S}})$.

4.2.1 Upper bound on profit for $\overline{\mathcal{S}}$

We first establish Condition 1 in the definition of a strong maxmin solution.

Proposition 1. *$\overline{\mathcal{S}}$ is a well-defined information structure. For all mechanisms $\mathcal{M}$ and equilibria β of $(\mathcal{M}, \overline{\mathcal{S}})$, $\Pi(\beta, \mathcal{M}, \overline{\mathcal{S}}) \leq \overline{\Pi}$.*

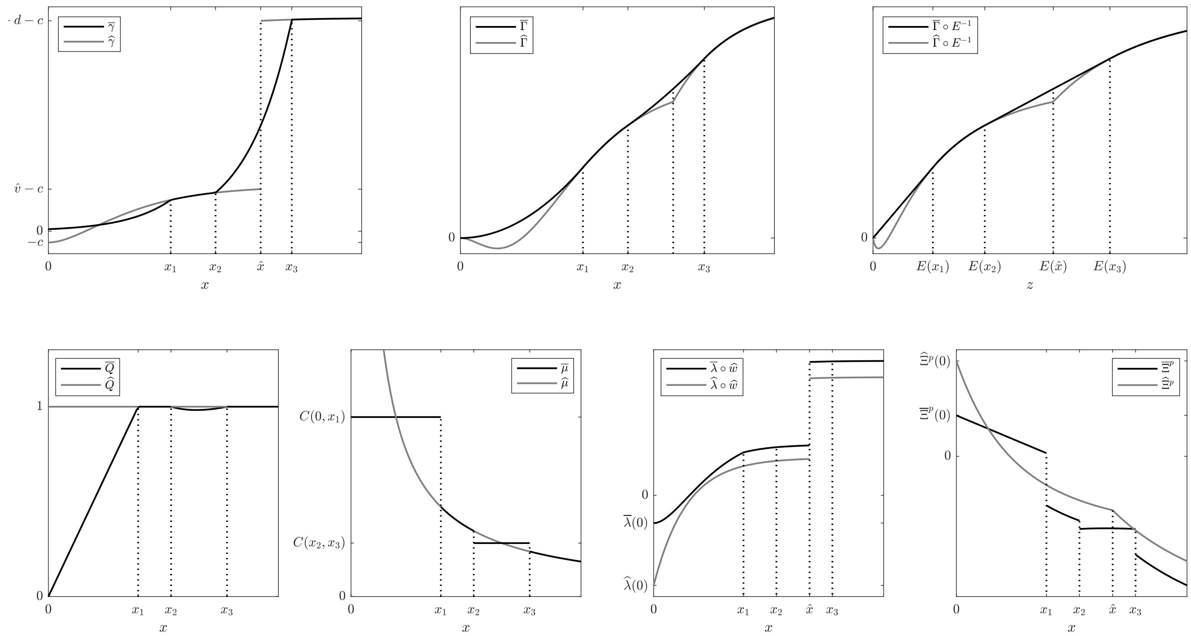

Figure 3: Objects used in the construction of the solution for $N = 2$, a value that is uniform on $[0, \widehat{v}] \cup [\widehat{v} + d, 1 + d]$ and a cost c. The cutoff $\widehat{x}$ is such that $G_2(\widehat{x}) = \widehat{v}$. The particular values used are $\widehat{v} = 0.95$, $d = 3$, and $c = 0.2$.

To prove Proposition 1, we need the following characterization of mean-preserving spreads.

Lemma 1. *Fix distributions F_1 and F_2. Then F_1 is a mean-preserving spread of F_2 if and only if for all $\alpha \in [0, 1]$,*

$$\int_{y=0}^{\alpha} \left(F_1^{-1}(y) - F_2^{-1}(y)\right) dy \geq 0, \tag{20}$$

with an equality at $\alpha = 1$.

This characterization reframes stochastic dominance in terms of ordering of conditional expectations. Specifically, F_1 is a mean-preserving spread of F_2 if for every $\alpha \in [0, 1]$, the average of the α lowest realizations under F_1 is less then than the average of the α lowest realizations under F_2. The proof of Lemma 1 is in Appendix A.

Now we argue that $\overline{\mathcal{S}}$ is in fact an information structure with respect to the value distribution H.

Lemma 2. *The value function $\overline{\gamma}$ is a well-defined and increasing function. H is a mean-preserving spread of the distribution of $\overline{w}(\Sigma s)$.*

Proof of Lemma 2. Since $\overline{\Gamma} \circ E^{-1}$ is a concave function, it is continuously differentiable at all but countably many points, and we can extend the derivative by right continuity.

Since E is also differentiable, we conclude that $\overline{\Gamma}$ has a right derivative as well. We can therefore define $\overline{\gamma}$ as specified.

We next argue that $\overline{\gamma}(x)$ is an increasing function. If x is such that there is an interval $[x, x+\epsilon)$ on which $\overline{\Gamma}$ coincides with $\widehat{\Gamma}$, then their right-derivatives at x must coincide as well, so that $\overline{\gamma}(x) = \widehat{\gamma}(x)$, where the latter is increasing. In addition, if $[a, b]$ is a graded interval and $x \in [a, b)$, then it must be that $\overline{\gamma}$ has an exponential shape, as

$$\frac{d}{dx}\overline{\Gamma}(x) = \frac{d}{dz}(\overline{\Gamma}(E^{-1}(z)))|_{z=E(x)}E'(x) = \frac{\widehat{\Gamma}(b) - \widehat{\Gamma}(a)}{b-a}\exp(x)g_N(x).$$

The constant is chosen so that $\overline{\Gamma}(x)$ coincides with $\widehat{\Gamma}(x)$ at the end points of the graded interval. Finally, note that $\overline{\gamma}$ cannot jump up at x, as this would create a convex kink in $\overline{\Gamma} \circ E^{-1}$ at $z = E(x)$, which contradicts the definition of $\overline{\Gamma} \circ E^{-1}$. Moreover, if $\overline{\gamma}$ jumped down at x, then this would have to happen at the end point of a graded interval at which $\overline{\Gamma}$ and $\widehat{\Gamma}$ coincide. Thus, both $\overline{\Gamma} \circ E^{-1}$ and $\widehat{\Gamma} \circ E^{-1}$ would have concave kinks at $z = E(x)$, which contradicts the fact that $\widehat{\gamma}$ is monotonically increasing. We conclude that $\overline{\gamma}$ is continuous at the end points of graded intervals, and hence is monotonically increasing.

We will next show that the distribution of $\widehat{\gamma}(\Sigma s)$ is a mean-preserving spread of a distribution of $\overline{\gamma}(\Sigma s)$. The lemma then follows from the observation that the distribution of $\widehat{\gamma}(\Sigma s) + c$ is H, and $\overline{w}(\Sigma s) = \overline{\gamma}(\Sigma s) + c$.

Let $\overline{F}$ and $\widehat{F}$ denote the cumulative distributions of $\overline{\gamma}(x)$ and $\widehat{\gamma}(x)$, respectively, where $x \sim G_N$. Since $\overline{\gamma}$ and $\widehat{\gamma}$ are both increasing, for all $\alpha \in [0, 1]$, $\overline{\gamma}(G_N^{-1}(\alpha)) = \overline{F}^{-1}(\alpha)$ and $\widehat{\gamma}(G_N^{-1}(\alpha)) = \widehat{F}^{-1}(\alpha)$. From a change of variables $x = G_N(y)$, we conclude that

$$\begin{aligned}\int_{y=0}^{\alpha}(\overline{F}^{-1}(y) - \widehat{F}^{-1}(y))dy &= \int_{x=0}^{G_N^{-1}(\alpha)}(\overline{\gamma}(x) - \widehat{\gamma}(x))g_N(x)dx \\ &= \overline{\Gamma}(G_N^{-1}(\alpha)) - \widehat{\Gamma}(G_N^{-1}(\alpha)) \geq 0\end{aligned}$$

from the definition of the concavification. Moreover, it must be that $\overline{\Gamma}(\infty) = \widehat{\Gamma}(\infty)$, since otherwise $\min\{\overline{\Gamma}(x), \widehat{\Gamma}(\infty)\} \circ E^{-1}$ would be a smaller concave function that dominates $\widehat{\Gamma} \circ E^{-1}$. The result then follows from Lemma 1. □

Next, we will need the following characterization of the graded gains function:

Lemma 3. *For all $x \in \mathbb{R}_+$ and $y \geq x$, $\overline{\gamma}(y) \leq \overline{\gamma}(x)\exp(y-x)$.*

Proof of Lemma 3. Since $\overline{\Gamma} \circ E^{-1}$ is concave, it must be that its derivative

$$\frac{\overline{\gamma}(E^{-1}(z))g_N(E^{-1}(z))}{E'(E^{-1}(z))} = \frac{\overline{\gamma}(E^{-1}(z))}{\exp(E^{-1}(z))}$$

is decreasing. As a result, the function $\overline{\gamma}(x)\exp(-x)$ is decreasing in x, which implies the result. □

We can now complete the proof of Proposition 1:

Proof of Proposition 1. From Lemma 2, we know that $\overline{\mathcal{S}}$ is well defined. To complete the proof, we now show that $\overline{\Pi}$ is an upper bound on profit.

Let us write

$$U_i(s_i, s_i') = \int_{\overline{S}_{-i}} (\overline{w}(s_i + \Sigma s_{-i})q_i(s_i', s_{-i}) - t_i(s_i', s_{-i})) \exp(-\Sigma s_{-i}) ds_{-i},$$

and $U_i(s_i) = U_i(s_i, s_i)$. If q_i is Nash implemented, then for all i, s_i, and s_i',

$$U_i(s_i) \geq U_i(s_i, s_i') = U_i(s_i') + \int_{\overline{S}_{-i}} (\overline{\gamma}(s_i + \Sigma s_{-i}) - \overline{\gamma}(s_i' + \Sigma s_{-i}))q_i(s_i', s_{-i}) \exp(-\Sigma s_{-i}) ds_{-i}.$$

Participation security also implies that $U_i(s_i) \geq 0$. Thus, for all $\Delta > 0$,

$$\begin{aligned}
U_i &= \int_{\overline{S}_i} U_i(s_i) \exp(-s_i) ds_i \\
&\geq \int_{\{s \in \overline{S} | s_i \geq \Delta\}} [U_i(s_i - \Delta) + (\overline{\gamma}(\Sigma s) - \overline{\gamma}(\Sigma s - \Delta))q_i(s_i - \Delta, s_{-i})] \exp(-\Sigma s) ds \\
&= \exp(-\Delta) \int_{\{s \in \overline{S} | s_i \geq \Delta\}} [U_i(s_i - \Delta) + (\overline{\gamma}(\Sigma s) - \overline{\gamma}(\Sigma s - \Delta))q_i(s_i - \Delta, s_{-i})] \exp(-(\Sigma s - \Delta)) ds \\
&= \exp(-\Delta) \left(U_i + \int_{\overline{S}} (\overline{\gamma}(\Sigma s + \Delta) - \overline{\gamma}(\Sigma s))q_i(s_i, s_{-i}) \exp(-\Sigma s) ds \right).
\end{aligned}$$

We claim that

$$\lim_{\Delta \to 0} \sup_{s_i \in [0, \Delta]} \int_{\overline{S}_{-i}} (\overline{\gamma}(s_i + \Sigma s_{-i} + \Delta) - \overline{\gamma}(s_i + \Sigma s_{-i})) \exp(-\Sigma s_{-i}) ds_{-i} = 0. \qquad (21)$$

To see this, note that

$$\sup_{s_i \in [0, \Delta]} \overline{\gamma}(s_i + \Delta + \Sigma s_{-i}) - \overline{\gamma}(s_i + \Sigma s_{-i}) \leq \overline{\gamma}(2\Delta + \Sigma s_{-i}) - \overline{\gamma}(\Sigma s_{-i}).$$

If $\overline{\gamma}$ is continuous at Σs_{-i}, then the right-hand side converges to zero as $\Delta \to 0$, so the left-hand side is squeezed to zero as well. Thus, the integrand in (21) converges to zero almost surely. Moreover, the integrand is bounded above by the integrable function $(\overline{v} - c) \exp(-\Sigma s_{-i})$. The claim then follows from the dominated convergence theorem.

Thus, for any $\epsilon > 0$, there exists a $\widehat{\Delta} > 0$ such that if $\Delta < \widehat{\Delta}$, then

$$U_i \geq \frac{1}{(\exp(\Delta) - 1)} \left[\int_{\overline{S}} (\overline{\gamma}(\Sigma s + \Delta) - \overline{\gamma}(\Sigma s))q_i(s_i, s_{-i}) \exp(-\Sigma s) ds - (1 - \exp(-\Delta))\epsilon \right].$$

Now, let

$$Q(x) = \frac{1}{g_N(x)} \int_{\{s \in \overline{S} | \Sigma s = x\}} \sum_{i=1}^{N} q_i(s) \exp(-\Sigma s) ds$$

be the expected probability of allocating the good conditional on the aggregate signal being x. Then clearly

$$\sum_{i=1}^{N} U_i \geq \frac{1}{(\exp(\Delta)-1)}\left[\int_{x=0}^{\infty}(\overline{\gamma}(x+\Delta)-\overline{\gamma}(x))Q(x)g_N(x)dx-(1-\exp(-\Delta))\epsilon\right].$$

Since total surplus is

$$\int_{x=0}^{\infty}\overline{\gamma}(x)Q(x)g_N(x)dx,$$

we conclude that an upper bound on profit is

$$\int_{x=0}^{\infty}\left[\overline{\gamma}(x)-\frac{\overline{\gamma}(x+\Delta)-\overline{\gamma}(x)}{\exp(\Delta)-1}\right]Q(x)g_N(x)dx+\exp(-\Delta)\epsilon.$$

Lemma 3 implies that the term multiplying $Q(x)$ is positive, and since $Q(x) \leq 1$, profit is bounded above by

$$\begin{aligned}&\int_{x=0}^{\infty}\left[\overline{\gamma}(x)-\frac{\overline{\gamma}(x+\Delta)-\overline{\gamma}(x)}{\exp(\Delta)-1}\right]g_N(x)dx+\exp(-\Delta)\epsilon\\ &=\int_{x=0}^{\infty}\overline{\gamma}(x)\left[g_N(x)+\frac{g_N(x)-g_N(x-\Delta)}{\exp(\Delta)-1}\right]dx+\exp(-\Delta)\epsilon,\end{aligned}$$

where $g_N(x)=0$ if $x<0$. The term in brackets converges pointwise for all positive x to $g_N(x)+g_N'(x)=g_{N-1}(x)$ as $\Delta \to 0$. To apply the dominated convergence theorem, all that remains is to present an integrable bounding function, which is done in Lemma 12 in Appendix A.

As a result, as $\Delta \to 0$, the profit bound converges to $\overline{\Pi}+\epsilon$. Since ϵ was arbitrary, we have the result. $\square$

This argument uses the same basic ideas as revenue equivalence theorem of Myerson (1981). We have, however, used the special structure of $\overline{\mathcal{S}}$ to skip a direct computation of virtual values and the integral representation for $U_i(s_i)$. In so doing, we have sidestepped a significant technical complication, since the value function $\overline{\gamma}$ need not be well-enough behaved for standard formulations of the envelope theorem to apply (cf. Milgrom and Segal, 2002).

4.2.2 Lower bound on profit for $\overline{\mathcal{M}}$

The next result establishes Condition 2 in the definition of a strong maxmin solution.

Proposition 2. *$\overline{\mathcal{M}}$ is a well-defined mechanism. For all information structures $\mathcal{S}$ and equilibria β of $(\overline{\mathcal{M}},\mathcal{S})$, $\Pi(\beta,\overline{\mathcal{M}},\mathcal{S}) \geq \overline{\Pi}$.*

We will repeatedly use the following result:

Lemma 4. *The aggregate allocation sensitivity $\overline{\mu}$ is decreasing. As a result, $\overline{\lambda}$ is concave.*

Proof of Lemma 4. On a non-graded interval, $\overline{\mu}(x) = (N-1)/x$, which is decreasing, and on a graded interval $[a,b]$, $\overline{\mu}(x) = C(a,b) \in [(N-1)/b, (N-1)/a]$. The fact that $\overline{\mu}$ is decreasing across graded intervals then follows from the definition of $C(a,b)$ and the well-known inequality

$$\frac{N-1}{N}\frac{1}{b}(b^N - a^N) \leq b^{N-1} - a^{N-1} \leq \frac{N-1}{N}\frac{1}{a}(b^N - a^N),$$

e.g., Hardy, Littlewood, and Pólya (1934, equation (2.15.2)).

Concavity of $\overline{\lambda}$ then follows from the fact $\overline{\mu}$ is decreasing and equation (19). □

As with the information structure, we next verify that $\overline{\mathcal{M}}$ is well-defined.

Lemma 5. *$\overline{\mathcal{M}}$ is a well-defined mechanism that satisfies participation security.*

Proof of Lemma 5. There are three critical properties that need to be verified. Feasibility of the allocation rule, existence of the transfers, and participation security.

The allocation rule is feasible as long as $\overline{Q}$ is between 0 and 1. Clearly it is non-negative since the constants $C(a,b)$ and $D(a,b)$ are positive. So it is sufficient to check that it is always less than 1. It is straightforward to argue that $\overline{Q}$ is equal to one at the end points of a graded interval. Moreover, the derivative of the allocation rule on a graded interval $[a,b]$ is

$$\overline{Q}'(x) = \frac{C(a,b)}{N} - (N-1)\frac{D(a,b)}{x^N}$$

which is increasing. So $\overline{Q}$ is convex on $[a,b]$, and therefore $\overline{Q}(x) \leq \max\{\overline{Q}(a), \overline{Q}(b)\} = 1$.[15]

Existence of the transfers comes down to arguing that the integrals in equations (17) and (19) are finite. First consider the last integral in (19), which is bounded above by

$$\int_{v=\underline{v}}^{\overline{v}} \overline{\mu}(G_N^{-1}(H(v))dv = \int_{y=0}^{\infty} \overline{\mu}(y)\widehat{w}(dy).$$

We will argue that this integral is finite using the left-tail condition (1), which implies that

$$\limsup_{x \to 0} \frac{\widehat{w}(x) - \underline{v}}{x} = 0,$$

Thus, there exist $C < \infty$ and $\widehat{x} > 0$ such that if $x < \widehat{x}$, $(\widehat{w}(x) - \underline{v})/x^{\varphi} \leq C$ for some $\varphi > 1$. If the value function is not graded at x, $\overline{\mu}(x) = (N-1)/x$, and if x is in a graded interval $[a,b]$, then

$$\overline{\mu}(x) = C(a,b) = \frac{b^N - ba^{N-1}}{b^N - a^N}\frac{N}{b} \leq \frac{N}{b} \leq \frac{N}{x}. \tag{22}$$

[15] We thank a referee for suggesting this simpler argument.

Thus, if $x \leq \widehat{x}$, we can plug in our bounds and integrate by parts to obtain

$$\begin{aligned}
\int_{y=0}^{\infty} \overline{\mu}(y)\widehat{w}(dy) &\leq \int_{y=0}^{\infty} \frac{N}{y}\widehat{w}(dy) \\
&= \int_{y=0}^{\widehat{x}} \frac{N}{y^2}(\widehat{w}(y) - \underline{v})dy + \int_{y=\widehat{x}}^{\infty} \frac{N}{y^2}(\widehat{w}(y) - \underline{v})dy \\
&\leq N\int_{y=0}^{\widehat{x}} Cy^{\varphi-2}dy + \int_{y=\widehat{x}}^{\infty} \frac{N}{y^2}(\overline{v} - \underline{v})dy \\
&= N\frac{1}{\varphi-1}C\widehat{x}^{\varphi-1} + N\frac{\overline{v} - \underline{v}}{\widehat{x}}.
\end{aligned}$$

We conclude that the last integral in the definition of $\overline{\lambda}$ is bounded. The middle integral is simply the expectation of the last integral across lower bounds $x \sim G_N$, so we conclude that $\overline{\lambda}$ is bounded.

Now, given that $\widehat{w}$ is bounded above and $\overline{\mu}$ is decreasing from Lemma 4, to show that $\overline{\Xi}^p$ is bounded, it is sufficient to show that $\limsup_{x\to 0}(\widehat{w}(x) - \underline{v})/x < \infty$. But as argued before this is a direct implication of (1). Hence, $\overline{t}_i^p$ is well defined and bounded.

Finally, participation security follows from the observation that $\overline{t}_i^p(0, m_{-i}) = 0$ for all m_{-i}, so that the ex post payoff from a message of 0 is $(v - \underline{v})\overline{q}_i(0, m_{-i}) \geq 0$. □

Next, we show that the allocation rule $\overline{q}$ is right-differentiable and has the aggregate allocation sensitivity in (16).

Lemma 6. *The allocation $\overline{q}_i(m)$ is right-differentiable with respect to m_i at every $m \neq 0$, and $\nabla \cdot \overline{q}(m) = \overline{\mu}(\Sigma m)$.*

Proof of Lemma (6). When m is such that $m_j = \infty$ for some j, then $\overline{q}_i(m_i + \epsilon, m_{-i}) = \overline{q}_i(m)$ for all ϵ, so $\overline{q}_i$ has a derivative of zero with respect to m_i at m. Thus, we have $\nabla \cdot \overline{q}(m) = 0 = \overline{\mu}(\infty)$.

Suppose $m_j < \infty$ for all j and $m \neq 0$. Since $\overline{q}_i(m_i, m_{-i}) = \frac{m_i}{\Sigma m}\overline{Q}(\Sigma m)$, for the right-differentiability of $\overline{q}_i$ it suffices to show that $\overline{Q}(x)$ is right differentiable at every $x > 0$. From the functional forms of $\overline{Q}$ and $\overline{\mu}$, it is easy to check that

$$\overline{Q}'(x) = \overline{\mu}(x) - \frac{N-1}{x}\overline{Q}(x) \tag{23}$$

for every x in the interior of a graded or non-graded interval. Since there are at most countably many graded intervals, for any $x' > x > 0$, we have

$$\overline{Q}(x') - \overline{Q}(x) = \int_{y=x}^{x'} \left(\overline{\mu}(y) - \frac{N-1}{y}\overline{Q}(y)\right) dy.$$

Since the absolute value of the above integrand is bounded by $2(N-1)/x$, $\overline{Q}$ is absolutely continuous on any interval $[a, b]$ with $a > 0$. Since $\overline{\mu}(y)$ is also right-continuous in y, $\overline{Q}(x)$ is right differentiable at every $x > 0$, and (23) holds for all $x > 0$.

Finally, it is easy to check that $\nabla \cdot \overline{q}(m) = \overline{\mu}(\Sigma m)$ using the product rule and equation (23). □

We now develop the lower bound on profit in $\overline{\mathcal{M}}$.

Lemma 7. *The unconditional expectation of $\overline{\lambda}(v)$ is $\overline{\Pi}$:*

$$\int_{v=\underline{v}}^{\overline{v}} \overline{\lambda}(v)H(dv) = \int_{x=0}^{\infty} \overline{\lambda}(\widehat{w}(x))g_N(x)dx = \overline{\Pi}.$$

Proof of Lemma 7. The equivalence of the two integrals follows from the change of variables $v = \widehat{w}(x) = H^{-1}(G_N(x))$. For the second equality, it is sufficient to show that the middle integral in (19) is equal to the unconditional expectation of the last integral, which follows from Tonelli's theorem:

$$\begin{aligned}\int_{x=0}^{\infty}\int_{y=x}^{\infty} \overline{\mu}(y)d\widehat{w}(y)g_N(x)dx &= \int_{x=0}^{\infty}\int_{y=0}^{x} g_N(y)dy\overline{\mu}(x)d\widehat{w}(x)\\ &= \int_{x=0}^{\infty} \overline{\mu}(x)G_N(x)d\widehat{w}(x).\end{aligned}$$

□

Lemma 8. *The unconditional expectation of $\overline{\Xi}^p$ is zero:*

$$\int_{x=0}^{\infty} \overline{\Xi}^p(x)g_N(x)dx = 0.$$

Proof of Lemma 8. Using the formula for $\overline{\Xi}^p$ in equation (18) and Lemma 7, it is sufficient to show that

$$\overline{\Pi} = \int_{x=0}^{\infty} \left(\overline{\mu}(x)(\widehat{w}(x) - \underline{v}) - (c - \underline{v})\overline{Q}(x)\right) g_N(x)\, dx. \tag{24}$$

Since $g_N'(x) = g_{N-1}(x) - g_N(x) = (N-1)g_N(x)/x - g_N(x)$, integration by parts gives:

$$\int_{x=0}^{\infty} \overline{Q}'(x)g_N(x)\, dx = \int_{x=0}^{\infty} \overline{Q}(x)\left(1 - \frac{N-1}{x}\right) g_N(x)\, dx,$$

which implies, using $\overline{\mu}(x) = \overline{Q}'(x) + \frac{N-1}{x}\overline{Q}(x)$,

$$\int_{x=0}^{\infty} \overline{Q}(x)g_N(x)\, dx = \int_{x=0}^{\infty} \left(\overline{Q}'(x) + \overline{Q}(x)\frac{N-1}{x}\right) g_N(x)\, dx = \int_{x=0}^{\infty} \overline{\mu}(x)g_N(x)\, dx.$$

Thus, since $\widehat{\gamma}(x) = \widehat{w}(x) - c$, applying the above equation and writing out the expression for $\overline{\Pi}$, we see that (24) is equivalent to

$$\int_{x=0}^{\infty} \overline{\gamma}(x)g_{N-1}(x)dx = \int_{x=0}^{\infty} \widehat{\gamma}(x)\overline{\mu}(x)g_N(x)dx.$$

When $\overline{\Gamma}(x) = \widehat{\Gamma}(x)$, we have $\overline{\gamma}(x) = \widehat{\gamma}(x)$ and $\overline{\mu}(x) = (N-1)/x$, so $\overline{\mu}(x)g_N(x) = g_{N-1}(x)$ and the two integrands above are exactly equal. On the other hand, over a graded interval

$[a, b]$,

$$\begin{aligned}\int_{x=a}^{b} \widehat{\gamma}(x)\overline{\mu}(x)g_N(x)dx &= C(a,b)(\widehat{\Gamma}(b) - \widehat{\Gamma}(a)) \\ &= C(a,b)(\overline{\Gamma}(b) - \overline{\Gamma}(a)) \\ &= C(a,b)\int_{x=a}^{b} \overline{\gamma}(x)g_N(x)dx \\ &= \int_{x=a}^{b} \overline{\gamma}(x)g_{N-1}(x)dx,\end{aligned}$$

since $\overline{\gamma}(x) = \overline{\gamma}(a)\exp(x-a)$ and for $n \geq 1$,

$$\int_{x=a}^{b} \overline{\gamma}(x)g_n(x)dx = \overline{\gamma}(a)\exp(-a)\frac{x^n}{n!}\bigg|_{x=a}^{b} = \overline{\gamma}(a)\exp(-a)\frac{b^n - a^n}{n!}.$$

□

Lemma 9. *The premium transfers are Lipschitz continuous in m_i uniformly across m_{-i}. Moreover, they satisfy for all m*

$$\nabla \cdot \overline{t}^p(m) - \Sigma\overline{t}^p(m) = \overline{\Xi}^p(\Sigma m),$$

where the derivative $\partial\overline{t}_i^p(m)/\partial m_i$ is defined by taking limits from the right.

Proof of Lemma 9. We can rewrite $\overline{t}_i^p$ as

$$\begin{aligned}\overline{t}_i^p(m) = &\frac{1}{N!}\sum_{\zeta\in Z}\int_{x=0}^{\infty}\overline{\Xi}^p(\Sigma m_{\zeta<\zeta(i)} + x)g_{N-\zeta(i)+1}(x)dx \\ &- \frac{1}{N!}\exp(m_i)\sum_{\zeta\in Z}\int_{y=m_i}^{\infty}\int_{x=0}^{\infty}\left(\overline{\Xi}^p(\Sigma m_{\zeta<\zeta(i)} + x + y\right)g_{N-\zeta(i)}(x)dx\exp(-y)dy.\end{aligned}$$

As a result, $\exp(-m_i)\overline{t}_i^p(m)$ is absolutely continuous and almost-everywhere differentiable and

$$\begin{aligned}\frac{\partial}{\partial m_i}\left(\exp(-m_i)\overline{t}_i^p(m)\right) = &-\exp(-m_i)\frac{1}{N!}\sum_{\zeta\in Z}\int_{x=0}^{\infty}\overline{\Xi}^p(\Sigma m_{\zeta<\zeta(i)} + x)g_{N-\zeta(i)+1}(x)dx \\ &+ \exp(-m_i)\frac{1}{N!}\sum_{\zeta\in Z}\int_{x=0}^{\infty}\overline{\Xi}^p\left(\Sigma m_{\zeta\leq\zeta(i)} + x\right)g_{N-\zeta(i)}(x)dx.\end{aligned}$$

Thus,

$$\frac{\partial\overline{t}_i^p(m)}{\partial m_i} = t_i(m) + \frac{1}{N!}\sum_{\zeta\in Z}\int_{x=0}^{\infty}\left[\overline{\Xi}^p(\Sigma m_{\zeta\leq\zeta(i)} + x)g_{N-\zeta(i)}(x) - \overline{\Xi}^p(\Sigma m_{\zeta<\zeta(i)} + x)g_{N-\zeta(i)+1}(x)\right]dx.$$

The above equation implies that $\frac{\partial \bar{t}_i^p(m)}{\partial m_i} = 0$ when $m_i = \infty$. Moreover, since $\bar{t}_i^p$ and $\overline{\Xi}^p$ are bounded, we conclude that $\bar{t}_i^p$ is Lipschitz continuous in m_i uniformly across m_{-i}.

Finally,

$$\begin{aligned}
&\nabla \cdot \bar{t}^p(m) - \Sigma \bar{t}^p(m) \\
&= \frac{1}{N!} \sum_{\zeta \in Z} \int_{x=0}^{\infty} \sum_{i=1}^{N} \left[\overline{\Xi}^p(\Sigma m_{\zeta \le \zeta(i)} + x) g_{N-\zeta(i)}(x) - \overline{\Xi}^p(\Sigma m_{\zeta < \zeta(i)} + x) g_{N-\zeta(i)+1}(x) \right] dx \\
&= \overline{\Xi}^p(\Sigma m) - \int_{x=0}^{\infty} \overline{\Xi}^p(x) g_N(x) dx,
\end{aligned}$$

so the result follows from Lemma 8. □

Lemma 10. *For any information structure $\mathcal{S}$ and equilibrium β of $(\overline{\mathcal{M}}, \mathcal{S})$, it must be that*

$$\int_S \int_{\overline{M}} ((w(s) - \underline{v}) \nabla \cdot \bar{q}(m) - \nabla \cdot \bar{t}^p(m)) \beta(dm|s) \pi(ds) \le 0, \tag{25}$$

where $\nabla \cdot \bar{q}(0) \equiv \bar{\mu}(0)$.

This intuitive result corresponds to the fact that any equilibrium, local upward deviations must not be attractive. If a bidder were to marginally increase all of the messages they send in in equilibrium, the change in payoff would be

$$\int_S \int_{\overline{M}} \left((w(s) - \underline{v}) \frac{\partial}{\partial m_i} \bar{q}_i(m) - \frac{\partial}{\partial m_i} \bar{t}_i^p(m) \right) \beta(dm|s) \pi(ds) \le 0.$$

Summing across i gives (25). A technical complication is that the allocation sensitivity may blow up as the aggregate message goes to zero. A rigorous proof is in Appendix A.

We can now complete the proof of Proposition 2.

Proof of Proposition 2. We have already argued in Lemma 5 that $\overline{\mathcal{M}}$ is well-defined. To complete the proof, it suffices to show that profit in any equilibrium in any information structure is at least $\overline{\Pi}$. This will be established in two steps.

Step 1: For any v and x,

$$\begin{aligned}
\bar{\lambda}(v) &= \bar{\lambda}(\widehat{w}(x)) - \int_{\nu=v}^{\widehat{w}(x)} \bar{\mu}(G_N^{-1}(H(\nu))) d\nu \\
&\le \bar{\lambda}(\widehat{w}(x)) - (\widehat{w}(x) - v)\bar{\mu}(x) \\
&= (\underline{v} - c)\overline{Q}(x) + (v - \underline{v})\bar{\mu}(x) - \overline{\Xi}^p(x),
\end{aligned}$$

where the second line follows from the fact that $\bar{\mu}$ is decreasing (Lemma 4), and the third line follows from the definition of $\overline{\Xi}^p$.

Step 2: Fix an information structure $\mathcal{S}$. Note that profit in an equilibrium β of $(\overline{\mathcal{M}}, \mathcal{S})$ is

$$\int_S \int_{\overline{M}} \left((\underline{v} - c)\overline{Q}(\Sigma m) + \Sigma \bar{t}^p(m) \right) \beta(dm|s) \pi(ds).$$

By Lemma 10 and Steps 1, this is at least

$$\begin{aligned}
&\int_S \int_{\overline{M}} \left((\underline{v}-c)\overline{Q}(\Sigma m) + (w(s)-\underline{v})\nabla \cdot \overline{q}(m) - (\nabla \cdot \overline{t}_i^p(m) - \Sigma \overline{t}^p(m))\right) \beta(dm|s)\pi(ds) \\
&= \int_S \int_{\overline{M}} \left((\underline{v}-c)\overline{Q}(\Sigma m) + (w(s)-\underline{v})\overline{\mu}(\Sigma m) - \overline{\Xi}^p(\Sigma m)\right) \beta(dm|s)\pi(ds) \\
&\geq \int_S \overline{\lambda}(w(s))\pi(ds) \\
&\geq \int_V \overline{\lambda}(v)H(dv).
\end{aligned}$$

The last line uses concavity of $\overline{\lambda}$ (Lemma 4), the fact that the distribution of $w(s)$ is a mean-preserving spread of H, and Jensen's inequality. The final integral is equal to $\overline{\Pi}$ by Lemma 7. □

4.2.3 Truth-telling equilibrium

We now come to the last condition for $(\overline{\mathcal{M}}, \overline{\mathcal{S}}, \overline{\beta})$ to be a strong maxmin solution.

Proposition 3. *The truthful strategies $\overline{\beta}$ are an equilibrium of the game $(\overline{\mathcal{M}}, \overline{\mathcal{S}})$.*

Proof of Proposition 3. Let

$$U_i(m_i, m_i') = \int_{\overline{M}_{-i}} \left((\overline{w}(m_i + \Sigma m_{-i}) - \underline{v})\overline{q}_i(m_i', m_{-i}) - \overline{t}_i^p(m_i', m_{-i})\right) \exp(-\Sigma m_{-i})\, dm_{-i}$$

denote the payoff from reporting m_i' when the true signal is m_i and when others report truthfully. Since the signal of $m_i = \infty$ occurs with probability zero in $\overline{\mathcal{S}}$, we can assume $m_i < \infty$.

From the definition of $\overline{t}_i^p$ and the fact that the unconditional expectation of $\overline{\Xi}^p$ is zero, we conclude that[16]

$$\int_{\overline{M}_{-i}} \overline{t}_i^p(m_i, m_{-i}) \exp(-\Sigma m_{-i}) dm_{-i} = -\int_{x=0}^{\infty} \overline{\Xi}^p(x + m_i) g_N(x) dx. \tag{26}$$

Next, note that when $\Sigma m_{-i} = x$,

$$\overline{q}_i(m_i', m_{-i}) = \frac{m_i'}{m_i' + x}\overline{Q}(m_i' + x) = \overline{Q}(m_i' + x) + \frac{x}{N-1}\overline{Q}'(m_i' + x) - \frac{x}{N-1}\overline{\mu}(m_i' + x).$$

[16]This equation substantiates the claim in Section 3 that (12) is equivalent to incentive compatibility of $\overline{q}$ on $\overline{\mathcal{S}}$. Using (11) and (12), we can rewrite the transfer as $t_i^p(m) = -\int_{x=0}^{\infty} \xi_i^p(m_i + x, m_{-i}) \exp(-x) dx$. But if we take the expectation of $\xi_i^p(m_i + x, m_{-i})$ over m_{-i}, equation(12), combined with (9), implies that the interim transfer is exactly (26). Moreover, when $N = 2$, the interim expected transfer can equal (26) for all m_i only if (12) holds as well. For more than two bidders, incentive compatibility of $\overline{q}$ on $\overline{\mathcal{S}}$ is equivalent to, for all i and m_i, $\int_{\overline{M}_{-i}} (\xi_i^p(m_i, m_{-i}) - \overline{\Xi}^p(m_i, m_{-i})) \exp(-\Sigma m_{-i}) dm_{-i} = 0$.

Integration by parts and the fact that $g_N'(x) = g_{N-1}(x) - g_N(x)$ yields the following:

$$
\begin{aligned}
&\int_{\overline{M}_{-i}} (\overline{w}(m_i + \Sigma m_{-i}) - \underline{v})\overline{q}_i(m_i', m_{-i}) \exp(-\Sigma m_{-i}) dm_{-i} \\
&= \int_{x=0}^{\infty} (\overline{w}(m_i + x) - \underline{v})(\overline{Q}(m_i' + x)g_{N-1}(x) - \overline{\mu}(m_i' + x)g_N(x))dx \\
&\quad + \int_{x=0}^{\infty} (\overline{w}(m_i + x) - \underline{v})\overline{Q}'(m_i' + x)g_N(x)dx \\
&= \int_{x=0}^{\infty} (\overline{w}(m_i + x) - \underline{v})(\overline{Q}(m_i' + x)g_{N-1}(x) - \overline{\mu}(m_i' + x)g_N(x))dx \\
&\quad - \int_{x=0}^{\infty} \overline{Q}(m_i' + x) \left[(\overline{w}(m_i + x) - \underline{v})(g_N(x) - g_{N-1}(x))dx - g_N(x)\overline{w}(m_i + dx)\right] \\
&= - \int_{x=0}^{\infty} (\overline{w}(m_i + x) - \underline{v})\overline{\mu}(m_i' + x)g_N(x)dx \\
&\quad - \int_{x=0}^{\infty} \overline{Q}(m_i' + x)g_N(x) \left[(\overline{w}(m_i + x) - \underline{v})dx - \overline{w}(m_i + dx)\right].
\end{aligned}
$$

Combining this expression with the one for interim transfers, and observing that $\overline{\gamma}(m_i + dx) = \overline{w}(m_i + dx)$, we conclude that

$$
\begin{aligned}
&U_i(m_i, m_i) - U_i(m_i, m_i') \\
&= \int_{x=0}^{\infty} (\overline{w}(m_i + x) - \underline{v})(\overline{\mu}(m_i' + x) - \overline{\mu}(m_i + x))g_N(x)dx \\
&\quad + \int_{x=0}^{\infty} (\overline{Q}(m_i + x) - \overline{Q}(m_i' + x))g_N(x) \left[(\overline{w}(m_i + x) - \underline{v})dx - \overline{w}(m_i + dx)\right] \\
&\quad + \int_{x=0}^{\infty} (\overline{\Xi}^p(x + m_i) - \overline{\Xi}^p(x + m_i'))g_N(x)dx \\
&= \int_{x=0}^{\infty} \left[(\overline{w}(m_i + x) - \overline{w}(m_i' + x))\overline{\mu}(m_i' + x) + \overline{\lambda}(\widehat{w}(m_i + x)) - \overline{\lambda}(\widehat{w}(m_i' + x))\right] g_N(x)dx \\
&\quad + \int_{x=0}^{\infty} (\overline{Q}(m_i + x) - \overline{Q}(m_i' + x))g_N(x) \left[\overline{\gamma}(m_i + x)dx - \overline{\gamma}(m_i + dx)\right].
\end{aligned}
$$

The integral in the second-to-last line is single peaked in m_i' with a peak at $m_i' = m_i$, since the integrand is equal to

$$
\overline{w}(m_i + x) - \overline{w}(m_i' + x))\overline{\mu}(m_i' + x) + \int_{y=m_i'+x}^{m_i+x} \overline{\mu}(y)\widehat{w}(dy) = - \int_{y=m_i'+x}^{m_i+x} \widehat{w}(y)\overline{\mu}(dy) \qquad (27)
$$

where we have integrated by parts. By Lemma 4, $\overline{\mu}$ is decreasing, so (27) crosses zero once from above. For the last integral in the expression for the deviation payoff, recall that by Lemma 3, $\overline{\gamma}(m_i + x) - d\overline{\gamma}(m_i + x)/dx \geq 0$, and it is zero on graded intervals. Thus, the integrand is non-zero only for those x at which the gains function is not graded at $m_i + x$, in which case $\overline{Q}(m_i + x) = 1 \geq \overline{Q}(m_i' + x)$. We conclude that the last integral as non-zero as well, so that $U_i(m_i, m_i) - U_i(m_i, m_i')$ is non-negative for all m_i', and truth telling is an equilibrium. □

Theorem 1 clearly follows from Propositions 1, 2, and 3.

4.3 The must-sell case

An important benchmark is the variant of our model where the Seller has to sell the good. All of our existing tools carry over to the must-sell setting and almost immediately give us the solution.

Let us define a *must-sell mechanism* to be one for which $Q(m) = 1$ for all m. A *must-sell strong maxmin solution* is a triple $(\mathcal{M}, \mathcal{S}, \beta)$ satisfying Conditions 1-3 in Section 2.5, but where $\mathcal{M}$ is a must-sell mechanism and Condition 2 only has to hold for $\mathcal{M}'$ that are must-sell mechanisms.

Now, consider the information structure $\widehat{\mathcal{S}}$ where the signals are i.i.d. and exponential draws on $\mathbb{R}_+$, and the value function is $\widehat{w}$, i.e., the fully-revealing value function. Also consider the mechanism $\widehat{\mathcal{M}}$ corresponding to the efficient proportional rule:

$$\widehat{q}_i(m) = \begin{cases} \frac{1}{N} & \text{if } \Sigma m = 0; \\ \frac{m_i}{\Sigma m} & \text{if } 0 < \Sigma m < \infty; \\ \frac{1}{|\{j|m_j=\infty\}|} & \text{if } \Sigma m = \infty. \end{cases}$$

We define $\widehat{\lambda}$, $\widehat{\Xi}^p$, $\widehat{t}_i$, and $\widehat{t}_i^p$ according to analogous formula as (16)–(19), using $\widehat{w}$ and $\widehat{\mu}(x) = (N-1)/x$ in place of $\overline{w}$ and $\overline{\mu}$. Let

$$\widehat{\Pi} = \int_{x=0}^{\infty} \widehat{\gamma}(x) g_{N-1}(x)\, dx. \tag{28}$$

Finally, let $\widehat{\beta}$ denote the same truthful/obedient strategies as were part of our general solution.

Theorem 2 (Must-sell solution)**.** *The triple $(\widehat{\mathcal{M}}, \widehat{\mathcal{S}}, \widehat{\beta})$ is a must-sell strong minmax solution with a profit guarantee of $\widehat{\Pi}$ defined by (28).*

Proof of Theorem 2. The proofs of Propositions 2 and 3 remain valid with $\widehat{\gamma}$ in place of $\overline{\gamma}$. Thus, the mechanism $\widehat{\mathcal{M}}$ guarantees the Seller at least $\widehat{\Pi}$ in any equilibrium, and $\widehat{\beta}$ is an equilibrium of the game $(\widehat{\mathcal{M}}, \widehat{\mathcal{S}})$.

The only place where our argument changes is in the proof of Proposition 1, where we had to use Lemma 3 to conclude that the profit upper bound is maximized by setting $Q(x) = 1$. In the must-sell case, this result is automatic. We therefore conclude that (28) is an upper bound on profit in $\widehat{\mathcal{S}}$. $\square$

We shall further explore welfare properties of must-sell mechanisms in Section 6.

4.4 The single-crossing case

We now discuss a class of distributions for which the maxmin mechanism is relatively simple and there is a natural interpretation of the allocation rule. We say a distribution

H is *single-crossing* if there is a cutoff $\overline{x}$ such that $\widehat{\Gamma} \circ E^{-1}$ is convex on $[0, \overline{x}]$ and is concave on $[\overline{x}, \infty]$. When the gains function is differentiable, this is equivalent to saying that $\widehat{\gamma}(x) - \widehat{\gamma}'(x)$, the virtual value, is single-crossing from below at $x = \overline{x}$. This is in a sense a counterpart to the regular case of Myerson (1981), where the Seller only has an incentive to ration the good when signals are a below a cutoff.

If the distribution is single crossing, then $\overline{\Gamma} \circ E^{-1}$ must be linear on $[0, z^*]$ and it coincides with $\widehat{\Gamma} \circ E^{-1}$ on $[z^*, \infty]$, for some $z^* \geq E(\overline{x})$. Setting $x^* = E^{-1}(z^*)$, the graded gains function is

$$\overline{\gamma}(x) = \begin{cases} \overline{\gamma}(x^*) \exp(x - x^*) & x < x^* \\ \widehat{\gamma}(x) & x \geq x^* \end{cases}.$$

As a result, on the graded interval $[0, x^*]$, we have $\overline{Q}(x) = xC(0, x^*)/N = x/x^*$, since $D(0, x^*) = 0$. The maxmin allocation is therefore

$$\overline{q}_i(m_i, m_{-i}) = \frac{m_i}{\max\{x^*, \Sigma m\}}.$$

We can interpret m_i as bidder i's nominal demand for the good and x^* as the unit of the demand, so m_i/x^* is the demand in units of probability of being allocated the good. With this interpretation, the allocation rule simply says that the bidders get their demands if the aggregate demand is feasible (i.e., less than 1), and if the aggregate demand is not feasible, then the good is rationed in proportion to the demands.

As an illustration, we now argue that the uniform distribution is single crossing for all N. The fully-revealing gains function is $\widehat{\gamma}(x) = G_N(x) - c$, which has a virtual value $G_N(x) - c - g_N(x)$. This is clearly zero when $x = 0$, and its derivative is

$$2g_N(x) - g_{N-1}(x) = \left(\frac{2x}{N-1} - 1\right) g_{N-1}(x),$$

so that the virtual value is decreasing for $x < (N-1)/2$ and increasing otherwise. This implies that $G_N(x) - c - g_N(x)$ crosses zero once, from below.

Thus, $\overline{\gamma}$ has an exponential shaped $\overline{\gamma}(0)\exp(x)$ on $[0, x^*]$, and is fully revealing above x^*. For these to meet smoothly, it must be that $\overline{\gamma}(0) = \exp(-x^*)(G_N(x^*) - c)$. Moreover, for the integrated gains functions to coincide at x^*, it must be that

$$\begin{aligned}\int_{x=0}^{x^*} (G_N(x) - c) g_N(x) dx = \frac{(G_N(x^*) - c)^2}{2} \\ &= \exp(-x^*)(G_N(x^*) - c) \int_{x=0}^{x^*} \exp(x) g_N(x) dx \\ &= (G_N(x^*) - c) g_{N+1}(x^*).\end{aligned}$$

Thus, the cutoff x^* (uniquely) solves

$$G_N(x^*) - c = 2g_{N+1}(x^*).$$

and maxmin profit is

$$\begin{aligned}\overline{\Pi} &= \int_{x=0}^{x^*} \overline{\gamma}(0)\exp(x) g_{N-1}(x)dx + \int_{x=x^*}^{\infty} (G_N(x) - c) g_{N-1}(x)dx \\ &= (G_N(x^*) - c) g_N(x^*) + \int_{x=x^*}^{\infty} (G_N(x) - c) g_{N-1}(x)dx,\end{aligned}$$

while maxmin profit among must-sell mechanisms is only

$$\widehat{\Pi} = \int_{x=0}^{\infty} (G_N(x) - c) g_{N-1}(x)dx.$$

These profit guarantees are compared for a range of N and for $c = 0$ in Figure 4, which is discussed in Section 6.

5 Uniqueness of the Value

We have constructed a particular strong maxmin solution. We now argue that while there may exist other solutions, as long as they are sufficiently well-behaved, they all must have the same profit guarantee, which is $\overline{\Pi}$.

Let us say that a mechanism is *finite* if the message spaces M_i are finite for all i. We can identify the finite message sets with with subsets of $\mathbb{N}$, so that the set of finite mechanisms, denoted by $\mathbf{M}^F$, is well-defined. Similarly, an information structure is *finite* if the signal spaces S_i are finite for all i, and the set of finite information structures is $\mathbf{S}^F$.

An information structure $\mathcal{S}$ is *regular* if for all $\mathcal{M} \in \mathbf{M}^F$, the game $(\mathcal{M}, \mathcal{S})$ has an equilibrium. A mechanism $\mathcal{M}$ is *regular* if for all $\mathcal{S} \in \mathbf{S}^F$, the game $(\mathcal{M}, \mathcal{S})$ has an equilibrium. A solution $(\mathcal{M}, \mathcal{S}, \beta)$ is *regular* if $\mathcal{M}$ and $\mathcal{S}$ are both regular.

Our first result for this section is the following:

Theorem 3 (Uniqueness)**.** *Every regular solution has a profit guarantee of* $\overline{\Pi}$*.*

The theorem follows from two propositions:

Proposition 4. *For all $\epsilon > 0$, there exists a finite mechanism $\mathcal{M}$ such that for every information structures $\mathcal{S}$ and equilibrium β of $(\mathcal{M}, \mathcal{S})$, expected profit is at least $\overline{\Pi} - \epsilon$.*

Proposition 5. *For all $\epsilon > 0$, there exists a finite information structure $\mathcal{S}$ such that for every mechanism $\mathcal{M}$ and equilibrium β of $(\mathcal{M}, \mathcal{S})$, expected profit is at most $\overline{\Pi} + \epsilon$.*

The proofs of these results are in Appendix B. As a function of ϵ, we construct a finite mechanism or a finite information structure for which the profit bound holds. The finite mechanisms are simply the restriction of $\overline{\mathcal{M}}$ to a grid of messages which are bounded away from zero. The profit lower bounds are computed via a discrete analogue of the weak duality argument of Proposition 2. As the lowest message and the space between messages converge to zero, and as the largest message goes to infinity, this bound converges to $\overline{\Pi}$. For the information structures, we approximate the limit by essentially drawing signals from $\overline{\mathcal{S}}$ and pooling signals which are in the same cell of a finite partition of $\overline{S}_i$. As the partition becomes finer, an upper bound on profit (obtained via a discretized form of the revenue equivalence theorem) converges to $\overline{\Pi}$.

The next question is whether our solution is regular. The following theorem answers in the affirmative.

Theorem 4 (Regularity)**.** *The solution $(\overline{\mathcal{M}}, \overline{\mathcal{S}}, \overline{\beta})$ is regular.*

The proof of this result is similarly relegated to Appendix B. For $\overline{\mathcal{S}}$, we simply verify that the conditions for equilibrium existence in Milgrom and Weber (1985) are satisfied, namely product-continuity of the signal distribution. For regularity of $\overline{\mathcal{M}}$, we verify that the sufficient conditions in Reny (1999) are satisfied for the normal form of $(\overline{\mathcal{M}}, \mathcal{S})$ whenever $\mathcal{S}$ is finite. This is relatively straightforward, because the only discontinuities in payoffs occur because the allocation rule jumps up in m_i at $m = 0$ or it jumps up in m_i when $m_i = \Sigma m_{-i} = \infty$. Only the first of these leads to a failure of lower semi-continuity of the allocation, and since transfers are continuous, bidders can protect themselves against this discontinuity by replacing the zero message with a small positive message.

Thus, all regular strong maxmin solutions have the same profit guarantee, and the strong maxmin solution we construct is itself regular. This justifies the styling of $\overline{\mathcal{M}}$ as a maxmin mechanism and of $\overline{\mathcal{S}}$ as a minmax information structure, which we now formalize as two corollaries.

Corollary 1. *Let $\mathbf{M}$ be a set of regular mechanisms which contains $\overline{\mathcal{M}}$, and fix a selection $\beta^*(\mathcal{M}, \mathcal{S})$ from the (non-empty) equilibrium correspondence B on $\mathbf{M} \times \mathbf{S}^F$. Then $\overline{\mathcal{M}}$ solves*

$$\max_{\mathcal{M} \in \mathbf{M}} \inf_{\mathcal{S} \in \mathbf{S}^F} \Pi(\beta^*(\mathcal{M}, \mathcal{S}), \mathcal{M}, \mathcal{S}).$$

Corollary 2. *Let $\mathbf{S}$ be a set of regular information structures which contains $\overline{\mathcal{S}}$, and fix a selection $\beta^*(\mathcal{M}, \mathcal{S})$ from the (non-empty) equilibrium correspondence B on $\mathbf{M}^F \times \mathbf{S}$. Then $\overline{\mathcal{S}}$ solves*

$$\min_{\mathcal{S} \in \mathbf{S}} \sup_{\mathcal{M} \in \mathbf{M}^F} \Pi(\beta^*(\mathcal{M}, \mathcal{S}), \mathcal{M}, \mathcal{S}).$$

These corollaries follow from the observations that $\overline{\Pi}$ bounds the solution to these problems (Propositions 4 and 5), and the bound is attained by $\overline{\mathcal{M}}$ and $\overline{\mathcal{S}}$ (Propositions 1 and 2).

We note that the arguments for Theorems 3 and 4 and Corollaries 1 and 2 are easily adapted to the must-sell case, without any qualifications. We comment further on this after the proofs in the Appendix.

6 Maxmin auctions in the many-bidder limit

6.1 Profit comparison

In this section, we will further explore the properties of the maxmin auction and the optimal profit guarantee. We begin with a comparison of mechanisms for the standard

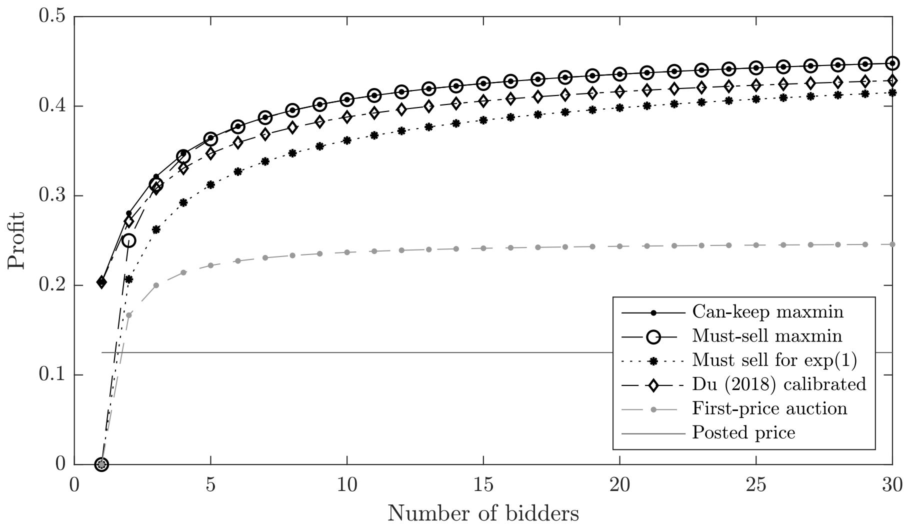

Figure 4: Comparing the maxmin mechanism to other auctions.

uniform distribution with $c = 0$. The optimal profit guarantees for this example was computed in Section 4.4. In Figure 4, we have plotted these optimal guarantees for N ranging from 1 to 30.[17] The can-keep and must-sell guarantees are the dots and circles, respectively.

For comparison, we have also plotted profit guarantee of the first-price auction, as computed by Bergemann, Brooks, and Morris (2017), which is the gray dots. This turns out to be $(N-1)/(4N-2)$ for the standard uniform distribution. We also plot as a solid black line the best guarantee from a posted price mechanism, which is $1/8$ and is obtained with a price of $1/4$.[18]

A striking feature of this picture is that the optimal profit guarantee increases in N and appears to be converging towards 0.5. The latter is the ex ante expected value, which is obviously an upper bound on profit in any mechanism. In fact, as N goes to infinity, the profit guarantee converges to the expected surplus. This remarkable fact is implied by the earlier result of Du (2018), who constructed a particular sequence of mechanisms and profit guarantees (the white diamonds) which converge to total surplus. A fortiori, the optimal profit guarantee must also converge to total surplus.

For the rest of this section, we explore and extend this result in a number of ways. We generalize the bound to the case where the Seller has a positive cost, and we argue that the correct limit profit is the ex ante gains from trade. We also characterize the rate at which the bound is attained, and we show that the limit is attained even with must-sell mechanisms. Finally, and perhaps most surprisingly, we will argue that asymptotic full surplus extraction holds even if the distribution of the value is misspecified. As an illustration, the black asterisks in Figure 4 are a profit guarantee for the maxmin auction

[17] A similar figure previously appeared in Du (2018).

[18] The worst-case information is when bidders learn publicly whether the value is below $1/2$. If it is below $1/2$, none of them buy, and if it is above, they strictly prefer to purchase.

that is calibrated to an exponentially distributed value but when the value is actually standard uniform.[19]

6.2 Information and welfare in the many-bidder limit

We now proceed formally. Before presenting our results, we have to address the left-tail condition on the value distribution introduced in Section 2. This assumption was only made for a single N, whereas now we will study the limit as N goes to infinity. It turns out, however, that no additional assumption is needed. The left-tail condition could equivalently be stated as

$$\limsup_{\alpha \to 0} \frac{H^{-1}(\alpha) - \underline{v}}{(G_N^{-1}(\alpha))^{\varphi}} < \infty,$$

for some $\varphi > 1$. But since G_N is decreasing in N in the first-order stochastic dominance order, G_N^{-1} is increasing, so that for all α and $N' > N$,

$$\frac{H^{-1}(\alpha) - \underline{v}}{(G_N^{-1}(\alpha))^{\varphi}} \geq \frac{H^{-1}(\alpha) - \underline{v}}{(G_{N'}^{-1}(\alpha))^{\varphi}},$$

so that the left-tail condition is satisfied for all $N' > N$ as well.

With this clarification out of the way, we can proceed with characterizing the limiting profit guarantee. We now denote the optimal profit guarantees for the can-keep and must-sell models by $\overline{\Pi}_N(H)$ and $\widehat{\Pi}_N(H)$, respectively, where we now emphasize their dependence on the number of bidders and the distribution. Implicitly, as we vary the number of bidders, we hold fixed the other parameters of the model, namely the distribution H and the cost c.

Note that a simple upper bound on the profit guarantee that holds for all N is the ex ante gains from trade. For it could be that the bidders have no information about the value at all, in which case the best the Seller can do is make the bidders a take-it-or-leave-it offer at a price equal to the ex ante expected value. The following proposition argues that this upper bound is tight:

Proposition 6 (Limiting profit guarantee)**.** *In the limit as N goes to infinity, the profit guarantees $\overline{\Pi}_N(H)$ and $\widehat{\Pi}_N(H)$ converge to the ex ante gains from trade at a rate of $1/\sqrt{N}$.*

The formal proof is non-trivial and is in Appendix C. We will here provide some intuition. Recall that at the minmax information structure, the aggregate signal is a sufficient statistic for the value. Since the signals are independent and identically distributed, the law of large numbers suggests that when N is large, there distribution of the aggregate

[19] Strictly speaking, we have assumed that the support of H is bounded, which is violated with the exponential distribution. In this calculation, we have taken the limit of the formulae for bounded distributions. We expect that our formal results can be extended to cover unbounded distributions for which the right tail is not too heavy, such as the exponential.

signal, appropriately scaled, is concentrated around the mean. To be more precise, we can change the units of each bidder's signal according to[20]

$$s_i^C = \frac{s_i - 1}{\sqrt{N}},$$

where the "C" denotes a central limit normalization. The centered aggregate signal is then

$$x = \Sigma s^C = \frac{\Sigma s - N}{\sqrt{N}}$$

which has cumulative distribution and probability density functions

$$G_N^C(x) = G_N(\sqrt{N}x + N)$$
$$g_N^C(x) = \sqrt{N} g_N(\sqrt{N}x + N),$$

respectively. We can correspondingly center the value function as $\overline{w}_N^C(x) = \overline{w}_N(\sqrt{N}x+N)$, etc, where we now emphasize the dependence of $\overline{w}$ and other objects on N.

With this normalization, the distribution of the aggregate signal will converge to a standard Normal with distribution and density denoted Φ and ϕ, respectively. We argue in Appendix C that the normalized fully-revealing gains function converges almost surely to $\widehat{\gamma}_\infty^C(x) = H^{-1}(\Phi(x))$, which is just a change of units from $\widehat{\gamma}_N$, and the graded gains function converges almost surely to

$$\overline{\gamma}_\infty^C(x) = \begin{cases} 0 & \text{if } x < x^*; \\ H^{-1}(\Phi(x)) & \text{if } x \geq x^*, \end{cases}$$

where x^* is the largest x such that

$$0 = \int_{y=-\infty}^{x} \widehat{\gamma}_\infty^C(y)\phi(y)dy.$$

Note that x^* will be $-\infty$ if $v - c > 0$ with probability one. Thus, in the limit, there is only grading at the bottom, and then only if there is positive probability that the gains from trade are non-positive.

At the same time, with the change of units, the hazard rate of each bidder's signal has changed from 1 to $\sqrt{N}$. When N is large, the bidders' virtual value will be approximately

$$\overline{\gamma}_\infty^C(x) - \frac{1}{\sqrt{N}}\frac{d}{dx}\overline{\gamma}_\infty^C(x).$$

In effect, each bidder's individual contribution to collective information about the value becomes vanishingly small as the number of bidders grows large. Each bidder's information rents must correspondingly go to zero as well. Since only the bidder who is allocated

[20]The discussion here uses the standard central limit normalization. In Appendix C, we use a different but asymptotically normalization where $s_i^c = (s_i - (N-1)/N)/\sqrt{N-1}$. This turns out to be much more analytically convenient, as in the proof of Lemma 16.

the good gets an information rent, we conclude that total bidder surplus goes to zero at a rate of $1/\sqrt{N}$. At the same time, it is always weakly optimal for the Seller to allocate the good, so that profit converges to the ex ante gains from trade.

This sketch glosses over some technical complications. The convergence of the gains function is only almost everywhere, and along the sequence of minmax information structures the hazard rate and the graded gains function are both changing. The formal proof deals with these issues by working directly with the integral for the difference between ex ante gains from trade and profit, scaled up by $\sqrt{N}$. We argue that this sequence converges to a positive constant, thus establishing the proposition.

6.3 Robustness to the prior

We have assumed that the Seller does not know the information structure but knows the value distribution exactly. There is a clear tension here. It turns out, however, that our results are robust to misspecification of the prior, as we now explain.

Let us suppose that the Seller runs the mechanism $\overline{\mathcal{M}}_N(H)$ that provides the optimal profit guarantee when the prior is H, where we now emphasize the dependence on both N and H. Let $\overline{\lambda}_N(v;H)$ denote the associated optimal dual multipliers given by (19). We established in the proof of Proposition 2 that a lower bound on profit is the expectation of $\overline{\lambda}_N(v;H)$. But in that dual argument, the prior H only appears at the last step as a mean-preserving spread of the distribution of $w(s)$. As a result, even if the prior is some $H' \neq H$, we still obtain a lower bound on profit, which is the expectation of $\overline{\lambda}_N(v;H)$ under H'. But since $\overline{\lambda}_N(v;H)$ is bounded and continuous, the change in the profit guarantee will be small as long as H and H' are close in the weak-* topology. This gives us the following result:

Proposition 7 (Profit guarantee for misspecified prior)**.** *In any equilibrium of $\overline{\mathcal{M}}_N(H)$ for any information structure where the value distribution is H', expected profit is bounded below by*

$$\overline{\Pi}_N(H,H') = \int_{v=0}^{\overline{v}} \overline{\lambda}_N(v;H)H'(dv),$$

which is a linear and weak-* continuous function of H'.

Proposition 7 says that when the prior is only slightly misspecified, the loss in the profit guarantee will be small. If the prior is badly misspecified, however, the loss can be substantial. But when the number of bidders is large, the loss from misspecification is vanishingly small and the profit guarantee will still be approximately the ex ante gains from trade, even if the prior is badly misspecified. This is formalized in the following result:

Proposition 8 (Prior-independent limiting profit guarantee)**.** *Suppose the support of H' is a subset of the convex hull of the support of H. In the limit as N goes to infinity, $\overline{\Pi}_N(H,H')$ converges to the ex ante gains from trade under H'.*

In the case when $c \leq \underline{v}$, this result is essentially a consequence of Propositions 6 and 7. To see why, consider what would happen if the Seller ran $\overline{\mathcal{M}}_N(H)$ but the true prior puts probability one on a particular value v. Proposition 7 says that profit must be at least $\overline{\lambda}_N(v;H)$. At the same time, profit in this counterfactual cannot be greater than $v-c$, which is the efficient surplus. But Proposition 6 says that expected profit guarantee under H converges to the ex ante gains from trade, which is only possible if $\overline{\lambda}_N(v;H)$ converges to $v-c$ H-almost surely.

This argument establishes Proposition 8 if H' is absolutely continuous with respect to H and there is common knowledge of gains from trade. The result is much stronger. In Appendix C, we show that $\overline{\lambda}_N(v)$ converges pointwise to $v-c$ for all v in the convex hull of the support of H, even on the boundary, and even when there is not common knowledge of gains from trade.

Note that while the profit guarantee converges to the ex ante gains from trade under the true prior H', that guarantee need not be positive. Thus, $\overline{\mathcal{M}}_N(H)$ may not be a maxmin mechanism in the many-bidder limit, in the event that the optimal profit guarantee is zero and it is better to shut down production entirely.

We also note that analogues of Propositions 7 and 8 also hold for the must-sell model. The necessary modifications to the proof are minor, which we explain after the proof in Appendix C.

To summarize, the maxmin mechanisms that are optimal for finite N are unimprovable for large N. This is true even if the Seller knows nothing about the value distribution beyond bounds on its support. While we have assumed that the support of H is bounded for technical convenience, we do not foresee any significant conceptual issues in extending our results as long as the right tail of H is not too heavy. Thus, if one uses the maxmin mechanism for a distribution with full support on $\mathbb{R}_+$, e.g., $H(v) = 1 - \exp(-v)$ as in Figure 4, pointwise convergence of the profit guarantee to ex ante gains from trade will hold for all H'.

7 Conclusion

This paper has studied the canonical auction design problem when values are common. The novelty is to use a robust criterion for measuring the performance of an auction. The spirit of the exercise is to identify mechanisms that are less vulnerable to misspecification of information and behavior and are therefore more viable for practical implementation, where a designer may be unwilling or unable to commit to a specific description of information.

The literature to which we contribute has previously shown that it is possible to obtain non-trivial profit guarantees across all information structures and equilibria, even with standard mechanisms like the first-price auction. It has also shown that there are mechanisms whose profit guarantees are unimprovable when the number of bidders is large. Our marginal contribution is to establish, in a rich class of environments, the precise limit of what can be attained. We have also developed new methodological insights for the characterization of maxmin mechanisms, namely the double revelation principle and the critical conditions on aggregate allocation sensitivity and the aggregate excess growth. In terms of the solution itself, we have shown that the optimal guarantee can be attained with mechanisms that have a simple one-dimensional bidding interface. The analysis also suggests that simple allocation rules, where bidders demand shares of the good and then receive allocations that proportional to their demands, can perform quite well. The transfer rules we derive are less transparent. But we feel that as a whole, the maxmin mechanisms are much closer to meeting practical constraints on implementation than are, say, mechanisms that ask the agents to report signals in an abstract information structure or to report their beliefs and higher-order beliefs.

To our knowledge, the auctions we construct are new to the literature. We are also unaware of any similar auctions which are used in practice. We therefore view our contribution as normative in nature. The advantage of our approach is that it stays within the Bayesian auction design framework, broadly defined, but allows us to remedy some conceptual and practical limitations associated with having to commit to a specific information structure.

To be sure, this modeling approach introduces new conceptual issues: Why should the bidders have common knowledge of the information structure, while the Seller does not? Why does the Seller not simply induce the bidders to reveal the information structure, and then run the optimal auction for whatever information structure they report? While this is clearly a theoretical possibility, such an approach runs contrary to the spirit of our exercise, which is to identify auctions with desirable welfare properties that will remain feasible when we respect both the designer and the agents' limited ability to articulate and communicate their beliefs and higher-order beliefs. We have not imposed such constraints explicitly in our model. But to us, the value of the model is not just in its assumptions, but also in the nature of the results: mechanisms that have desirable welfare properties but are also low-dimensional, so that there is hope that they will remain feasible even if additional practical constraints are imposed. As for the common prior among the agents, this assumption is obviously controversial, but we find it relatively palatable as an as-if description of agents' behavior, as long as it does not need to be explicitly input into the mechanism design black box in order to compute the optimal mechanism.

Nonetheless, it is true that in distancing ourselves from the untenable knowledge assumptions of the standard model, we have taken an equally extreme position, which is that the designer puts no restrictions on information except for the prior on the value and the existence of a common prior. Verily, the truth must lie somewhere in between. Designers may be willing to rule out some models without committing themselves to a single description of the world. We expect the theory to become even more useful as we explore the middle ground between these two extremes, by incorporating reasonable restrictions on beliefs into the robust mechanism design problem.

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A Omitted Proofs for Section 4

Proof of Lemma 1. Recall the characterization of mean-preserving spreads in terms of orderings of integrated cumulative distributions in equation (2). We will show that (20) holds for all α if and only if (2) holds for all x. Define the graph

$$\mathrm{Gr}(F) = \{(x, \alpha) : x \geq 0, \alpha \in [F(x^-), F(x)]\}$$

for a cumulative distribution F, where $F(x^-)$ is the left-limit of F at x.

Note that for any $(x, \alpha) \in \mathrm{Gr}(F_1)$,

$$\int_{y=0}^{x} F_1(y)\, dy = x \cdot \alpha - \int_{y=0}^{\alpha} F_1^{-1}(y)\, dy,$$

and similarly for F_2. As a result, if $\mathrm{Gr}(F_1)$ and $\mathrm{Gr}(F_2)$ cross each other at (x, α), i.e., $(x, \alpha) \in \mathrm{Gr}(F_1) \cap \mathrm{Gr}(F_2)$, then (20) and (2) are equivalent.

Now consider an interval $(\underline{x}, \overline{x})$ where $F_1(x^-) > F_2(x)$ for all $x \in (\underline{x}, \overline{x})$, and

$$[F_1(\underline{x}^-), F_1(\underline{x})] \cap [F_2(\underline{x}^-), F_2(\underline{x})] = [\underline{\alpha}', \underline{\alpha}] \neq \emptyset,$$
$$[F_1(\overline{x}^-), F_1(\overline{x})] \cap [F_2(\overline{x}^-), F_2(\overline{x})] = [\overline{\alpha}', \overline{\alpha}] \neq \emptyset.$$

Then $F_1^{-1}(\alpha) > F_2^{-1}(\alpha)$ for all $\alpha \in (\underline{\alpha}, \overline{\alpha}')$. Thus, the integrals on the left-hand sides of (20) and (2) are strictly increasing in $x \in (\underline{x}, \overline{x})$ and $\alpha \in (\underline{\alpha}, \overline{\alpha}')$, respectively. Therefore, if (20) holds at $\alpha = \underline{\alpha}$, then (20) holds every $\alpha \in (\underline{\alpha}, \overline{\alpha}')$; and if (2) holds at $x = \underline{x}$, (2) holds for every $x \in (\underline{x}, \overline{x})$.

The case where $F_1(x) < F_2(x^-)$ on $(\underline{x}, \overline{x})$ is analogous and is omitted. □

To prove Lemma 10, we need the following technical result:

Lemma 11. *For all x, $|\overline{Q}'(x)| \leq (N-1)/x$ and if $\Sigma m = x$, then*

$$\left| \frac{1}{\Delta} \left(\overline{q}_i(m_i + \Delta, m_{-i}) - \overline{q}_i(m) \right) \right| \leq \frac{N+1}{x}.$$

Proof of Lemma 11. Note that

$$\overline{Q}'(x) = \frac{C(a,b)}{N} - \frac{(N-1)D(a,b)}{x^N}.$$

From equation (22), this is at most $1/x$ when we replace $C(a,b)$ with the bound in equation (22) and set $D(a,b) = 0$. Moreover, since $C(a,b) \geq 0$ and

$$D(a,b) = \frac{b^N - ab^{N-1}}{b^N - a^N} a^{N-1} \leq a^{N-1} \leq x^{N-1}, \tag{29}$$

we conclude that $\overline{Q}'$ is at least $-(N-1)/x$, so $|\overline{Q}'(x)| \leq (N-1)/x$.

Next, observe that

$$
\begin{aligned}
&\left|\frac{1}{\Delta}\left(q_i(m_i+\Delta,m_{-i})-q_i(m)\right)\right| \\
&=\left|\frac{1}{\Delta}\left(\frac{m_i+\Delta}{\Sigma m+\Delta}\overline{Q}(\Sigma m+\Delta)-\frac{m_i}{\Sigma m}\overline{Q}(m)\right)\right| \\
&=\left|\frac{1}{\Delta}\left(\frac{m_i}{\Sigma m+\Delta}(\overline{Q}(\Sigma m+\Delta)-\overline{Q}(\Sigma m))+\frac{\Delta}{\Sigma m+\Delta}\overline{Q}(\Sigma m+\Delta)+\left(\frac{m_i}{\Sigma m+\Delta}-\frac{m_i}{\Sigma m}\right)\overline{Q}(m)\right)\right| \\
&\leq\left|\frac{m_i}{\Delta}\frac{\overline{Q}(\Sigma m+\Delta)-\overline{Q}(m)}{\Sigma m+\Delta}\right|+\left|\frac{\overline{Q}(\Sigma m+\Delta)}{\Sigma m+\Delta}\right|+\left|\frac{m_i}{\Sigma m(\Sigma m+\Delta)}\overline{Q}(\Sigma m)\right| \\
&\leq\frac{m_i}{\Delta}\frac{\Delta\frac{N-1}{x}}{\Sigma m+\Delta}+\frac{2}{\Sigma m}leq\frac{N+1}{x}.
\end{aligned}
$$

where the last line follows from the facts that $\overline{Q}'(x)\leq(N-1)/x$ and $\overline{Q}(x)\leq 1$. □

Proof of Lemma 10. Fix an information structure $\mathcal{S}$ and an equilibrium β of $(\overline{\mathcal{M}},\mathcal{S})$. Since strategies are an equilibrium, for all i and $\Delta>0$,

$$
\int_S\int_{\overline{M}}\left[(w(s)-\underline{v})\frac{\overline{q}_i(m_i+\Delta,m_{-i})-\overline{q}_i(m)}{\Delta}-\frac{\overline{t}_i^p(m_i+\Delta,m_{-i})-\overline{t}_i^p(m)}{\Delta}\right]\beta(dm|s)\pi(ds)\leq 0. \tag{30}
$$

The following argument essentially sums this equation across i and takes a particular sequence $\Delta_k\to 0$ to argue that the inequality remains valid when we replace the discrete differences with the corresponding divergences.

Let us first consider the allocation. For every $x>0$, Lemma 11 implies that if $\Sigma m>x$, $(\overline{q}_i(m_i+\Delta,m_{-i})-\overline{q}_i(m))/\Delta$ is bounded below by $-(N+1)/x$. Let us define $\overline{M}^{x+}=\{m\in\overline{M}|\Sigma m>x\}$. Fatou's Lemma therefore implies that for every $\epsilon>0$, there is a $\widehat{\Delta}>0$ such that if $\Delta<\widehat{\Delta}$,

$$
\begin{aligned}
&\int_S\int_{\overline{M}^{x+}}(w(s)-\underline{v})\sum_{i=1}^N\frac{\overline{q}_i(m_i+\Delta,m_{-i})-\overline{q}_i(m)}{\Delta}\beta(dm|s)\pi(ds)+\epsilon \\
&\geq\liminf_{\Delta'\to 0}\int_S\int_{\overline{M}^{x+}}(w(s)-\underline{v})\sum_{i=1}^N\frac{\overline{q}_i(m_i+\Delta',m_{-i})-\overline{q}_i(m)}{\Delta'}\beta(dm|s)\pi(ds) \\
&\geq\int_S\int_{\overline{M}^{x+}}(w(s)-\underline{v})\nabla\cdot\overline{q}(m)\beta(dm|s)\pi(ds),
\end{aligned}
$$

for every $\Delta\leq\widehat{\Delta}$.

As $x\to 0$, the non-negative integrand $\mathbb{I}_{\Sigma m>x}(v-\underline{v})\nabla\cdot\overline{q}(m)$ converges monotonically, so the monotone convergence theorem implies that

$$
\begin{aligned}
&\lim_{x\to 0}\int_S\int_{\overline{M}^{x+}}(w(s)-\underline{v})\nabla\cdot\overline{q}(m)\beta(dm|s)\pi(ds) \\
&=\int_S\int_{\overline{M}^{0+}}(w(s)-\underline{v})\nabla\cdot\overline{q}(m)\beta(dm|s)\pi(ds).
\end{aligned}
$$

On the other hand, the integrand $\left(\sum_{i=1}^{N} \mathbb{I}_{\Sigma m > x} (\overline{q}_i(m_i + \Delta) - \overline{q}_i(m))/\Delta\right) \beta(dm|s)\pi(ds)$ is bounded by the integrable function $N/\Delta\beta(dm|s)\pi(ds)$. The dominated convergence theorem then implies that

$$\begin{aligned}
&\lim_{x\to 0} \int_S \int_{\overline{M}^{x+}} (w(s) - \underline{v}) \sum_{i=1}^{N} \frac{\overline{q}_i(m_i + \Delta, m_{-i}) - \overline{q}_i(m)}{\Delta} \beta(dm|s)\pi(ds) \\
&= \int_S \int_{\overline{M}^{0+}} (w(s) - \underline{v}) \sum_{i=1}^{N} \frac{\overline{q}_i(m_i + \Delta, m_{-i}) - \overline{q}_i(m)}{\Delta} \beta(dm|s)\pi(ds)
\end{aligned}$$

We conclude that for every $\epsilon > 0$, there exists a $\widehat{\Delta} > 0$ such that for every $\Delta \leq \widehat{\Delta}$,

$$\begin{aligned}
&\int_S \int_{\overline{M}^{0+}} (w(s) - \underline{v}) \sum_{i=1}^{N} \frac{\overline{q}_i(m_i + \Delta, m_{-i}) - \overline{q}_i(m)}{\Delta} \beta(dm|s)\pi(ds) + \epsilon \\
&\geq \int_S \int_{\overline{M}^{0+}} (w(s) - \underline{v}) \nabla \cdot \overline{q}(m) \beta(dm|s)\pi(ds).
\end{aligned} \tag{31}$$

Next, if there is a graded interval of the form $[0, b]$, $b > 0$, then any sequence $\Delta_k \to 0$ satisfies

$$\lim_{k\to\infty} \sum_{i=1}^{N} \frac{\overline{q}_i(\Delta_k, 0) - \overline{q}_i(0)}{\Delta_k} = \frac{N}{b} = \nabla \cdot \overline{q}(0).$$

Otherwise, if there is no graded interval at 0, then we can find a sequence $\Delta_k \to 0$ such that the gains function is not graded at Δ_k. Thus, $\overline{Q}(\Delta_k) = 1 = \overline{q}_i(\Delta_k, 0)$, so that

$$\lim_{k\to\infty} \sum_{i=1}^{N} \frac{\overline{q}_i(\Delta_k, 0) - \overline{q}_i(0)}{\Delta_k} = \infty = \nabla \cdot \overline{q}(0).$$

Letting $\overline{M}^0 = \{m \in \overline{M} | \Sigma m = 0\}$, we therefore have

$$\begin{aligned}
&\int_S \int_{\overline{M}^0} (w(s) - \underline{v}) \nabla \cdot \overline{q}(m) \beta(dm|s)\pi(ds) \\
&= \lim_{k\to\infty} \int_S \int_{\overline{M}^0} (w(s) - \underline{v}) \sum_{i=1}^{N} \frac{\overline{q}_i(\Delta_k, 0) - \overline{q}_i(0)}{\Delta_k} \beta(dm|s)\pi(ds).
\end{aligned}$$

In addition, equation (31) implies that

$$\begin{aligned}
&\int_S \int_{\overline{M}^{0+}} (w(s) - \underline{v}) \nabla \cdot \overline{q}(m) \beta(dm|s)\pi(ds) \\
&\leq \liminf_{k\to\infty} \int_S \int_{\overline{M}^{0+}} (w(s) - \underline{v}) \sum_{i=1}^{N} \frac{\overline{q}_i(m_i + \Delta_k, m_{-i}) - \overline{q}_i(m)}{\Delta_k} \beta(dm|s)\pi(ds).
\end{aligned}$$

Putting these two together, we conclude that

$$\begin{aligned}&\int_S \int_{\overline{M}} (w(s) - \underline{v}) \nabla \cdot \overline{q}(m) \beta(dm|s)\pi(ds) \\ &\leq \liminf_{k\to\infty} \int_S \int_{\overline{M}} (w(s) - \underline{v}) \sum_{i=1}^{N} \frac{\overline{q}_i(m_i + \Delta_k, m_{-i}) - \overline{q}_i(m)}{\Delta_k} \beta(dm|s)\pi(ds).\end{aligned} \tag{32}$$

Next, from Lemma 9, we know that $\partial \overline{t}_i^p / \partial m_i$ and $(\overline{t}_i^p(m_i + \Delta_k, m_{-i}) - \overline{t}_i^p(m))/\Delta_k$ are both bounded. The dominated convergence theorem then implies that

$$\begin{aligned}&\lim_{k\to\infty} \sum_{i=1}^{N} \int_S \int_{\overline{M}} \frac{\overline{t}_i^p(m_i + \Delta_k, m_{-i}) - \overline{t}_i^p(m)}{\Delta_k} \, \beta(dm|s)\pi(ds) \\ &= \int_S \int_{\overline{M}} \nabla \cdot \overline{t}_i^p(m) \, \beta(dm|s)\pi(ds).\end{aligned} \tag{33}$$

The lemma then follows from taking the limit of of (30) evaluated at Δ_k as $k \to \infty$, using (32) and (33). $\square$

Lemma 12. *There exists a $\widehat{\Delta}$ such that for all $\Delta < \widehat{\Delta}$ and $x \in \mathbb{R}_+$,*

$$\overline{\gamma}(x) \left[g_N(x) + \frac{g_N(x) - g_N(x - \Delta)}{\exp(\Delta) - 1} \right] \leq \begin{cases} \overline{\gamma}(x)(g_N(x) + 1) & \text{if } x < 1; \\ \overline{\gamma}(x)[g_N(x) + 2\overline{v} g_{N-1}(x)] & \text{if } x \geq 1. \end{cases}$$

This bounding function is integrable.

Proof of Lemma 12. If $x \geq \Delta$,

$$\begin{aligned}\frac{g_N(x) - g_N(x - \Delta)}{\exp(\Delta) - 1} &= \frac{\exp(-x)}{(N-1)!} \frac{x^{N-1} - (x - \Delta)^{N-1} \exp(\Delta)}{\exp(\Delta) - 1} \\ &\leq \frac{\exp(-x)}{(N-1)!} \frac{x^{N-1} - (x - \Delta)^{N-1}}{\exp(\Delta) - 1} \\ &\leq \frac{\exp(-x)}{(N-1)!} (N-1) x^{N-2} \frac{\Delta}{\exp(\Delta) - 1} \\ &= g_{N-1}(x) \frac{\Delta}{\exp(\Delta) - 1},\end{aligned}$$

where the third to last line follows from convexity of x^{N-1}, so $x^{N-1} - (x - \Delta)^{N-1} \leq (N-1)x^{N-2}\Delta$. Since $\Delta/(\exp(\Delta) - 1) \to 1$ as $\Delta \to 0$, we can take $\widehat{\Delta}$ small enough so that for $\Delta < \widehat{\Delta}$, the ratio is less than 2. Also, as long as $\Delta < N - 1$, g_N is increasing for $x \in [0, \Delta]$, so that for x in this range

$$\frac{g_N(x) - g_N(x - \Delta)}{\exp(\Delta) - 1} \leq \frac{g_N(\Delta)}{\exp(\Delta) - 1}$$

which converges to zero pointwise.

Integrability follows from the fact that $\overline{\gamma}(x)$ is bounded by $\overline{v}$. $\square$

B Proofs for Uniqueness and Regularity

B.1 Proof of Proposition 4

Before proving Proposition 4, we establish two technical results. Define the functions

$$h_n^\zeta(m) = \int_{x=0}^{\infty} \overline{\Xi}^p(\Sigma m_{\zeta\leq n} + x) g_{N-n+1}(x)dx$$
$$\tilde{h}_n^\zeta(m) = \int_{x=0}^{\infty} \overline{\Xi}^p(\Sigma m_{\zeta\leq n} + x) g_{N-n}(x)dx$$

Lemma 13. *For every $n \leq N$, $h_n^\zeta(m)$ is Lipschitz continuous in m_i, uniformly across m_{-i}. If $n < N$, then $\tilde{h}_n^\zeta(m)$ is Lipschitz in m_i, uniformly across m_{-i}, as well.*

Proof of Lemma 13. First, if $\zeta(i) > n$ then $h_n^\zeta(m)$ is invariant to m_i. Otherwise,

$$\begin{aligned} h_n^\zeta(m) &= \int_{x=0}^{\infty} \overline{\Xi}^p(\Sigma m_{\zeta\leq n} + x) g_{N-n+1}(x)dx \\ &= \exp(m_i) \int_{y=m_i}^{\infty} \int_{x=0}^{\infty} \overline{\Xi}^p(\Sigma m_{\zeta(-i)\leq n} + x + y) G_{N-n}(dx) \exp(-y)dy \end{aligned}$$

where G_0 is a Dirac measure on 0. Thus,

$$\frac{\partial}{\partial m_i}(\exp(-m_i)h_n^\zeta(m)) = -\exp(-m_i) \int_{x=0}^{\infty} \overline{\Xi}^p(\Sigma m_{\zeta\leq n} + x) G_{N-n}(dx)$$

so

$$\frac{\partial}{\partial m_i} h_n^\zeta(m) = \int_{x=0}^{\infty} \overline{\Xi}^p(\Sigma m_{\zeta\leq n} + x)(g_{N-n+1}(x)dx - G_{N-n}(dx)).$$

Clearly the right-hand side is bounded above by $2L$, where L is a bound on $\overline{\Xi}^p$, which proves the result.

Finally, all of the arguments are the same with $\tilde{h}_n^\zeta$ as long as $N - n > 0$. □

Lemma 14. *If $m_i \geq \overline{m}$, then*

$$\frac{\partial}{\partial m_i} \bar{t}_i^p(m) \geq -\frac{N-1}{\overline{m}}.$$

Proof of Lemma 14. Note that if $\overline{\Xi}^p(x)$ is increasing, then it is because $\overline{Q}(x)$ is decreasing, which has a derivative bounded by $(N-1)/\overline{m}$ from Lemma 11. A simple integration by parts, together with the fact that $g_{N-n+1}(0) = 0$, yields

$$\begin{aligned} \frac{\partial}{\partial m_i} h_n^\zeta(m) &= \int_{x=0}^{\infty} g_{N-n}(x) \overline{\Xi}^p(\Sigma m_{\zeta\leq n} + dx) \\ &\leq \int_{x=0}^{\infty} g_{N-n}(x) \frac{N-1}{\overline{m}} dx = \frac{N-1}{\overline{m}}. \end{aligned}$$

Thus,

$$\frac{\partial}{\partial m_i}\overline{t}_i^p(m) = -\frac{1}{N!}\sum_{\zeta\in Z}\frac{\partial}{\partial m_i}h^{\zeta}_{\zeta(i)}(m) \geq -\frac{N-1}{\overline{m}}.$$

□

Proof of Proposition 4. We will use the following discrete mechanism $\mathcal{M}$, which implicitly depends on parameters $\underline{m} > 0$ and $\Delta > 0$. We choose $\Delta = 1/\sqrt{K}$ for some positive integer K, and the message space is

$$M_i = \{\underline{m}, \underline{m} + \Delta, \ldots, \underline{m} + K\Delta\}.$$

Note that the highest message $\overline{m}$ is at least Δ^{-1}. We retain the base payment of $\underline{v}q_i(m)$, the premium is

$$t_i^p(m) = \overline{t}_i^p(m) - \overline{t}_i^p(\underline{m}, m_{-i}) + \overline{t}_i^p(0, m_{-i}).$$

and the allocation is $q_i(m) = \overline{q}_i(m)$. It is clear that this is a well-defined mechanism and that participation security is satisfied.

The discrete aggregate allocation sensitivity is

$$\mu(m) = \frac{1}{\Delta}\sum_{i=1}^{N}\mathbb{I}_{m_i<\overline{m}}(q_i(m_i + \Delta, m_{-i}) - q_i(m)),$$

and the discrete premium total excess growth is

$$\Xi^p(m) = \frac{1}{\Delta}\sum_{i=1}^{N}\mathbb{I}_{m_i<\overline{m}}(t_i(m_i + \Delta, m_{-i}) - t_i(m)) - \Sigma t(m).$$

Now, define

$$\lambda(m; v) = (v - \underline{v})\mu(m) - \Xi^p(m) + (\underline{v} - c)\overline{Q}(\Sigma m),$$

and let $\lambda(v) = \min_{m\in M}\lambda(m; v)$.

Next, we will use the following result:

Claim 1. *For all information structures $\mathcal{S}$ and equilibria β of $(\mathcal{S}, \mathcal{M})$, expected profit is at least $\int_v \lambda(v)H(dv)$.*

Proof of Claim 1. The equilibrium hypothesis implies that for all i,

$$\int_S \sum_{m\in M}\big[(w(s) - \underline{v})(q_i(\min\{m_i + \Delta, \overline{m}\}, m_{-i}) - q_i(m)) \\ - (t_i(\min\{m_i + \Delta, \overline{m}\}, m_{-i}) - t_i(m))\big]\beta(m|s)\pi(ds) \leq 0,$$

which corresponds incentive constraint for deviating to $\min\{m_i+\Delta, \overline{m}\}$. Summing across bidders, and dividing by Δ, we conclude that

$$\int_S \sum_{m\in M} ((w(s)-\underline{v})\mu(m) - \Xi^p(m) - \Sigma t^p(m))\beta(m|s)\pi(ds) \le 0.$$

Hence, expected profit is

$$\begin{aligned}
&\int_S \sum_{m\in M} \left(\Sigma t^p(m) + (\underline{v}-c)Q(\Sigma m)\right)\beta(m|s)\pi(ds)\\
&\ge \int_S \sum_{m\in M} \left(\Sigma t^p(m) + (\underline{v}-c)Q(\Sigma m) + (w(s)-\underline{v})\mu(m) - \Xi^p(m) - \Sigma t^p(m))\right)\beta(m|s)\pi(ds)\\
&= \int_S \sum_{m\in M} \left((w(s)-\underline{v})\mu(m) - \Xi^p(m) + (\underline{v}-c)Q(\Sigma m)\right)\beta(m|s)\pi(ds)\\
&\ge \int_S \lambda(w(s))\pi(ds)\\
&\ge \int_V \lambda(v)H(dv),
\end{aligned}$$

where the last line follows from the mean-preserving spread condition on $w(s)$ and that λ is concave, being the minimum of linear functions. $\square$

The rest of the argument shows that there exists an $\underline{m} > 0$ and $\Delta > 0$ such that $\lambda(v) \ge \overline{\lambda}(v) - \epsilon$ for all v, which will prove the proposition.

Claim 2. *For all $m \in M$,*

$$\begin{aligned}
\mu(m) \ge \frac{1}{\Delta}\int_{y=0}^{\Delta} \overline{\mu}(\Sigma m + y)dy - \frac{N+1}{\overline{m}}\\
- \frac{N(N-1)}{\Delta}\left(\log(N\underline{m}+\Delta) + \frac{N\underline{m}}{N\underline{m}+\Delta} - \log(N\underline{m}) - 1\right).
\end{aligned}$$

For fixed $\underline{m} > 0$, all terms on the right-hand side except the integral go to zero as $\Delta \to 0$.

Proof of Claim 2. From Lemma 11, we know that

$$\begin{aligned}
\mu(m) &= \sum_{i=1}^{N} \frac{1}{\Delta}(q_i(m_i+\Delta, m_{-i}) - q_i(m)) - \sum_{i=1}^{N} \mathbb{I}_{m_i=\overline{m}}\frac{1}{\Delta}(q_i(m_i+\Delta, m_{-i}) - q_i(m))\\
&\ge \sum_{i=1}^{N} \frac{1}{\Delta}(q_i(m_i+\Delta, m_{-i}) - q_i(m)) - \frac{N+1}{\overline{m}}.
\end{aligned}$$

x Recall that

$$\overline{\mu}(x) = \frac{N-1}{x}\overline{Q}(x) + \overline{Q}'(x).$$

Also recall that

$$\frac{\partial q_i(m_i)}{\partial m_i} = \frac{\Sigma m_{-i}}{(\Sigma m)^2}\overline{Q}(\Sigma m) + \frac{m_i}{\Sigma m}\overline{Q}'(\Sigma m).$$

Thus,

$$\begin{aligned}
&\sum_{i=1}^{N} \frac{1}{\Delta}\left(q_i(m_i+\Delta, m_{-i}) - q_i(m)\right) \\
&= \frac{1}{\Delta}\sum_{i=1}^{N}\int_{y=0}^{\Delta} \frac{\partial q_i(m_i+y, m_{-i})}{\partial m_i} dy \\
&= \frac{1}{\Delta}\sum_{i=1}^{N}\int_{y=0}^{\Delta} \left[\frac{\Sigma m_{-i}}{(\Sigma m+y)^2}\overline{Q}(\Sigma m+y) + \frac{m_i+y}{\Sigma m+y}\overline{Q}'(\Sigma m+y)\right] dy \\
&= \frac{1}{\Delta}\int_{y=0}^{\Delta} \left[\frac{(N-1)\Sigma m}{(\Sigma m+y)^2}\overline{Q}(\Sigma m+y) + \frac{\Sigma m+Ny}{\Sigma m+y}\overline{Q}'(\Sigma m+y)\right] dy \\
&= \frac{1}{\Delta}\int_{y=0}^{\Delta} \overline{\mu}(\Sigma m+y)dy - \frac{N-1}{\Delta}\int_{y=0}^{\Delta}\frac{y}{\Sigma m+y}\left[\frac{\overline{Q}(\Sigma m+y)}{\Sigma m+y} - \overline{Q}'(\Sigma m+y)\right] dy.
\end{aligned}$$

We need to bound the last integral from below. If x is in a non-graded interval, then $\overline{Q}(x)/x - \overline{Q}'(x)$ is just $1/x$. If x is in a graded interval $[a,b]$, then

$$\frac{\overline{Q}(x)}{x} - \overline{Q}'(x) = \frac{C(a,b)}{N} + \frac{D(a,b)}{x^N} - \frac{C(a,b)}{N} + (N-1)\frac{D(a,b)}{x^N} = \frac{ND(a,b)}{x^N}.$$

From equation (29), $D(a,b) \geq x^N$, so that the integrand in this case is at most N/x, and

$$\begin{aligned}
\int_{y=0}^{\Delta}\frac{y}{x+y}\left[\frac{\overline{Q}(x+y)}{x+y} - \overline{Q}'(x+y)\right] dy &\leq N\int_{y=0}^{\Delta}\frac{y}{(x+y)^2}dy \\
&= \int_{y=0}^{\Delta}\left(\frac{1}{x+y} - \frac{x}{(x+y)^2}\right) dy \\
&= N\left(\log(x+\Delta) + \frac{x}{x+\Delta} - \log(x) - 1\right).
\end{aligned}$$

The derivative with respect to x is

$$\frac{1}{x+\Delta} - \frac{1}{x} + \frac{\Delta}{(x+\Delta)^2} = \Delta\left(\frac{1}{(x+\Delta)^2} - \frac{1}{x(x+\Delta)}\right)$$

which is clearly negative, so subject to $x \geq N\underline{m}$, the expression is maximized with $x = N\underline{m}$, which gives us the lower bound on μ.

Moreover, as $\Delta \to 0$, $(N+1)/\overline{m} \to 0$, and by L'Hôpital's rule,

$$\lim_{\Delta\to 0}\left(\frac{\log(N\underline{m}+\Delta) + \frac{N\underline{m}}{N\underline{m}+\Delta} - \log(N\underline{m}) - 1}{\Delta}\right) = \lim_{\Delta\to 0}\left(\frac{1}{N\underline{m}+\Delta} - \frac{N\underline{m}}{(N\underline{m}+\Delta)^2}\right) = 0.$$

This completes the proof. $\square$

Claim 3. *Let L be a Lipschitz constant for $\overline{t}_i^p(m)$ and $h_n^\zeta(m)$. Then*

$$\Xi^p(x) \leq \frac{1}{\Delta}\int_{y=0}^{\Delta} \overline{\Xi}^p(\Sigma m + y)dy + N\left((3\Delta + \underline{m})L + (N-1)\Delta\right).$$

Proof of Claim 3. Let e_i denote the vector which has a one in the ith coordinate and zero everywhere else. We can rewrite the discrete difference in the premium as

$$\begin{aligned}
t_i^p(m_i+\Delta, m_{-i}) - t_i^p(m) &= -\frac{1}{N!}\sum_{\zeta\in Z}(h_{\zeta(i)}^\zeta(m+e_i\Delta) - h_{\zeta(i)}^\zeta(m)) \\
&= -\frac{1}{N!}\sum_{\zeta\in Z}\int_{y=0}^{\Delta}\frac{\partial}{\partial m_i}h_{\zeta(i)}^\zeta(m+e_iy)dy \\
&= -\frac{1}{N!}\sum_{\zeta\in Z}\int_{y=0}^{\Delta}(h_{\zeta(i)}^\zeta(m+e_iy) - \tilde{h}_{\zeta(i)}^\zeta(m+e_iy))dy \\
&\leq -\frac{1}{N!}\sum_{\zeta\in Z}\int_{y=0}^{\Delta}(h_{\zeta(i)}^\zeta(m+y/N) - \tilde{h}_{\zeta(i)}^\zeta(m+y/N))dy + 2L\Delta^2 \\
&= -\int_{y=0}^{\Delta}\frac{\partial}{\partial m_i}\overline{t}_i^p(m+y/N)dy + 2L\Delta^2
\end{aligned}$$

where we have used the fact that $h_{\zeta(i)}^\zeta$ and $\tilde{h}_{\zeta(i)}^\zeta$ are Lipschitz continuous, the latter when $\zeta(i) < N$, and $\tilde{h}_N^\zeta(m+e_iy) = \tilde{h}_N^\zeta(m+y/N) = \overline{\Xi}^p(\Sigma m + y)$. Also, Lipschitz continuity of $\overline{t}_i^p$ gives

$$\left|t_i^p(m) - \frac{1}{\Delta}\int_{y=0}^{\Delta}\overline{t}_i^p(m_i+y, m_{-i})dy\right| \leq L\Delta + \underline{m}L.$$

In addition, from Lemma 14 we can conclude that if $m_i = \overline{m} \geq \Delta^{-1}$,

$$\frac{1}{\Delta}\int_{y=0}^{\Delta}\frac{\partial}{\partial m_i}\overline{t}_i^p(m+y/N)dy + (N-1)\Delta \geq 0.$$

Combining these expressions, we conclude that

$$\Xi^p(m) \leq \sum_{i=1}^{N}\left[\frac{1}{\Delta}\int_{y=0}^{\Delta}\left(\frac{\partial}{\partial m_i}\overline{t}_i^p(m+y/N) - \overline{t}_i^p(m+y/N)\right)dy + (3\Delta+\underline{m})L + (N-1)\Delta\right]$$

which is the bound in the statement of the claim. □

We can now prove the proposition. We first argue that there exists a $\Delta > 0$ such that $\lambda(m;v) \geq \overline{\lambda}(m;v) - \epsilon$ for all $m \in M$ and $v \in [\underline{v}, \overline{v}]$, where

$$\overline{\lambda}(m;v) = (v-\underline{v})\overline{\mu}(\Sigma m) - \overline{\Xi}^p(\Sigma m) + (\underline{v}-c)\overline{Q}(\Sigma m).$$

From Lemma 11, we know that $|\overline{Q}(x+y) - \overline{Q}(x)| \leq y(N-1)/\underline{m}$. Thus,

$$\left|\overline{Q}(x) - \frac{1}{\Delta}\int_{y=0}^{\Delta} \overline{Q}(x+y)dy\right| \leq \frac{1}{\Delta}\int_{y=0}^{\Delta} \left|\overline{Q}(x+y) - Q(x)\right| dy$$
$$\leq \frac{1}{\Delta}\int_{y=0}^{\Delta} y\frac{N-1}{\underline{m}}dy = \Delta\frac{N-1}{2\underline{m}}.$$

Combining this inequality with and Claims 2, and 3, we get that

$$\begin{aligned}
\lambda(m;v) &= (v-\underline{v})\mu(m) - \Xi^p(m) + (\underline{v}-c)\overline{Q}(\Sigma m) \\
&\geq \frac{1}{\Delta}\int_{y=0}^{\Delta} \big((v-\underline{v})\overline{\mu}(\Sigma m + y) - \overline{\Xi}^p(\Sigma m + y) + (\underline{v}-c)\overline{Q}(\Sigma m + y)\big)\, dy \\
&\quad - N\left((3\Delta + \underline{m})L + (N-1)\Delta\right) \\
&\quad - \overline{v}\frac{N(N-1)}{\Delta}\left(\log(N\underline{m}+\Delta) + \frac{N\underline{m}}{N\underline{m}+\Delta} - \log(N\underline{m}) - 1\right) - \overline{v}\Delta\frac{N-1}{2\underline{m}} \\
&\geq \inf_{\{m' | \Sigma m \leq \Sigma m' \leq \Sigma m + y\}} \overline{\lambda}(m';v) \\
&\quad - N\left((3\Delta + \underline{m})L + (N-1)\Delta\right) \\
&\quad - \overline{v}\frac{N(N-1)}{\Delta}\left(\log(N\underline{m}+\Delta) + \frac{N\underline{m}}{N\underline{m}+\Delta} - \log(N\underline{m}) - 1\right) - \overline{v}\Delta\frac{N-1}{2\underline{m}},
\end{aligned}$$

where L is the Lipschitz constant for $\bar{t}_i(m)$. We can first pick $\underline{m} > 0$ so that $N\underline{m}L < \epsilon/2$. Since for fixed $\underline{m}$, the remaining terms in the last two lines go to zero as $\Delta \to 0$, we can pick a Δ small enough such that they sum to less than $\epsilon/2$. We then conclude that

$$\lambda(m;v) \geq \inf_{m' \in \mathbb{R}^+_N} \overline{\lambda}(m';v) - \epsilon = \overline{\lambda}(v) - \epsilon.$$

Hence, $\lambda(v) \geq \overline{\lambda}(v) - \epsilon$, and Claim 1 and Lemma 7 give the result. □

This proof goes through verbatim with the maxmin must-sell mechanism $\widehat{\mathcal{M}}$.

B.2 Proof of Proposition 5

Proof of Proposition 5. Fix $\Delta = 1/\sqrt{K}$ where K is a positive integer. We will later choose K sufficiently large, and equivalently Δ sufficiently small, to attain the desired ϵ. We define the information structure $\mathcal{S}$ as follows. The bidders get independent signals in

$$S_i = \{0, \Delta, \ldots, K\Delta\}.$$

Note that the highest message is just Δ^-. The probability mass function of s_i is

$$f_i(s_i) = \begin{cases} (1-\exp(-\Delta))\exp(-s_i) & \text{if } s_i < \Delta^{-1}; \\ \exp(-\Delta^{-1}) & \text{if } s_i = \Delta^{-1}. \end{cases}$$

As a result, s_i/Δ is a censored geometric random variable with arrival rate $1-\exp(-\Delta)$. We will write $f(s) = \times_{i=1}^N f_i(s_i)$ for the joint probability, and

$$F_i(s_i) = \sum_{s_i' \leq s_i} f_i(s_i') = \begin{cases} 1-\exp(-s_i-\Delta) & \text{if } s_i < \Delta^{-1}; \\ 1 & \text{otherwise,} \end{cases}$$

for the cumulative distribution. The value function is

$$w(s) = \frac{1}{f(s)} \int_{\left\{s' \in \mathbb{R}_+^N \middle| \tau(s_i') = s_i \forall i\right\}} \overline{w}(\Sigma s') \exp(-\Sigma s') ds,$$

where

$$\tau(x) = \begin{cases} \Delta \lfloor x/\Delta \rfloor & \text{if } x < \Delta^{-1}; \\ \Delta^{-1} & \text{otherwise.} \end{cases}$$

An interpretation is that we draw "true" signals s' for the bidders from $\overline{\mathcal{S}}$ and agent i observes $s_i = \min\{\Delta\lfloor \Delta^{-1} s_i' \rfloor, \Delta^{-1}\}$, i.e., signals above Δ^{-1} are censored and otherwise they are rounded down to the nearest multiple of Δ, and w is the conditional expectation of $\overline{w}$ given the profile of noisy observations s. Thus, the distribution of $\overline{w}$ is a mean-preserving spread of the distribution of w, so that H is a mean-preserving spread of the distribution of w as well.

Claim 4. *If $s_i < \Delta^{-1}$ for all i, then $w(s)$ only depends on the sum of the signals $l = \Sigma s$ and*

$$w(s) = \frac{\exp(l)}{(1-\exp(-\Delta))^N} \int_{x=l}^{l+N\Delta} \overline{w}(x) \rho(x-l) \exp(-x) dx,$$

and $\rho(y)$ is the $N-1$-dimensional volume of the set $\{s \in [0,\Delta]^N | \Sigma s = y\}$.

Proof of Claim 4. First observe that

$$f(s) = (1-\exp(-\Delta))^N \exp(-\Sigma s) = (1-\exp(-\Delta))^N \exp(-l).$$

Thus,

$$\begin{aligned} w(s) &= \frac{\exp(l)}{(1-\exp(-\Delta))^N} \int_{\left\{s \in \mathbb{R}_+^N | \tau(s) = t\right\}} \overline{w}(\Sigma s) \exp(-\Sigma s) ds \\ &= \frac{\exp(l)}{(1-\exp(-\Delta))^N} \int_{x=l}^{l+N\Delta} \int_{\left\{s \in \mathbb{R}_+^N | \tau(s) = t, \Sigma s = x\right\}} \overline{w}(\Sigma s) \exp(-\Sigma s) ds dx \\ &= \frac{\exp(l)}{(1-\exp(-\Delta))^N} \int_{x=l}^{l+N\Delta} \overline{w}(x) \exp(-x) \int_{\left\{s \in \mathbb{R}_+^N | \tau(s-t) = 0, \Sigma s = x\right\}} ds dx \\ &= \frac{\exp(l)}{(1-\exp(-\Delta))^N} \int_{x=l}^{l+N\Delta} \overline{w}(x) \exp(-x) \int_{\left\{s \in \mathbb{R}_+^N | \tau(s) = 0, \Sigma s = x-l\right\}} ds dx, \end{aligned}$$

where the inner integral is just $\rho(x-l)$. $\square$

We will now abuse notation slightly by writing $w(l)$ for the value when $l = \Sigma S$, and let $\gamma(l) = w(l) - c$.

Claim 5. *If $l > \Delta$, then $\gamma(l) \leq \exp(\Delta)\gamma(l - \Delta)$.*

Proof of Claim 5. From the definition of w, we know that

$$\begin{aligned}
\gamma(l) &= \frac{\exp(l)}{(1-\exp(-\Delta))^N} \int_{x=l}^{l+N\Delta} \overline{\gamma}(x)\exp(-x)\rho(x-l)dx \\
&= \frac{\exp(l)}{(1-\exp(-\Delta))^N} \int_{x=l-\Delta}^{l+(N-1)\Delta} \overline{\gamma}(x+\Delta)\exp(-x-\Delta)\rho(x-l+\Delta)dx \\
&\leq \frac{\exp(l-\Delta)}{(1-\exp(-\Delta))^N} \int_{x=l-\Delta}^{l+(N-1)\Delta} \overline{\gamma}(x)\exp(\Delta)\exp(-x)\rho(x-l+\Delta)dx \\
&= \exp(\Delta)\gamma(l-\Delta),
\end{aligned}$$

where the inequality follows from Lemma 3. □

Claim 6. *If the direct allocation $q_i(s)$ is Nash implemented by a participation secure mechanism, profit is at most*

$$\sum_{s\in S} f(s) \sum_{i=1}^{N} q_i(s) \left[\gamma(\Sigma s) - \frac{1-F_i(s_i)}{f_i(s_i)}(\gamma(\Sigma s+\Delta) - \gamma(\Sigma s))\right]. \tag{34}$$

Proof of Claim 6. This follows from standard revenue equivalence arguments: If we write $U_i(s_i, s_i')$ for the utility of a signal s_i that reports s_i', with $U_i(s_i) = U_i(s_i, s_i)$, then

$$U_i(s_i) \geq U_i(s_i, s_i') = U_i(s_i') + \sum_{s_{-i}\in S_{-i}} f_{-i}(s_{-i})q_i(s_i', s_{-i})\left(\gamma(s_i + \Sigma s_{-i}) - \gamma(s_i' + \Sigma s_{-i})\right).$$

Thus

$$U_i(s_i) = U_i(0) + \sum_{k=0}^{s_i/\Delta - 1} \sum_{s_{-i}\in S_{-i}} f_{-i}(s_{-i})q_i(k\Delta, s_{-i})\left(\gamma((k+1)\Delta + \Sigma s_{-i}) - \gamma(k\Delta + \Sigma s_{-i})\right).$$

The expectation of $U_i(s_i)$ across s_i is therefore

$$\begin{aligned}
&\sum_{s\in S} f(s) \sum_{k=0}^{s_i/\Delta} q_i(k\Delta, s_{-i})\left(\gamma((k+1)\Delta + \Sigma s_{-i}) - \gamma(k\Delta + \Sigma s_{-i})\right) \\
&= \sum_{s\in S} f(s)q_i(s)(\gamma(\Sigma s + \Delta) - \gamma(\Sigma s))\frac{1-F_i(s_i)}{f_i(s_i)}.
\end{aligned}$$

The formula then follow from subtracting the bound on aggregate bidder rents from total surplus. □

Let Π denote the maximum of the profit bound (34) across all q_i. Let $\tilde{\Pi}$ denote the profit bound when we set $q_1(s) \equiv 1$ and $q_j(s) \equiv 0$ for all $j \neq 1$.

Claim 7. $\Pi \leq \tilde{\Pi} + (1 - (1 - \exp(-\Delta))^N)\overline{v}$.

Proof of Claim 7. When signals are all less than Δ^{-1}, the bidder-independent virtual value is

$$\begin{aligned}
&\gamma(l) - \frac{1}{\exp(\Delta) - 1}\left(\gamma(l + \Delta) - \gamma(l)\right) \\
&\geq \gamma(l) - \frac{\exp(-\Delta)}{1 - \exp(-\Delta)}(\gamma(l)\exp(\Delta) - \gamma(l)) = 0,
\end{aligned}$$

where the inequality follows from Claim 5. Thus, the virtual value is maximized by allocating with probability one to bidder 1. With probability $1 - (1 - \exp(-\Delta^{-1}))^N$, one of the signals is above Δ^{-1}, in which case $\overline{v}$ is an upper bound on the virtual value. $\square$

Claim 8. $\lim_{\Delta\to 0} \tilde{\Pi} \leq \overline{\Pi}$.

Proof of Claim 8. Plugging in $q_1 \equiv 1$, we find that

$$\begin{aligned}
\tilde{\Pi} &= \sum_{s_{-1}\in S_{-1}} f_{-1}(s_{-1}) \sum_{s_1 \in S_1} \left[f_1(s_1)\gamma(\Sigma s) - \sum_{s_1' > s_1} f_1(s_1')(\gamma(\Sigma s + \Delta) - \gamma(\Sigma s)) \right] \\
&= \sum_{s_{-1}\in S_{-1}} f_{-1}(s_{-1}) \sum_{s_1 \in S_1} \left[f_1(s_1) \left[\gamma(\Sigma s) + \sum_{s_1' < s_1} (\gamma(s_1' + \Sigma s_{-1}) - \gamma(s_1' + \Sigma s_{-1} + \Delta)) \right]\right] \\
&= \sum_{s_{-1}\in S_{-1}} f_{-1}(s_{-1})\gamma(\Sigma s_{-1}).
\end{aligned}$$

Using the definition of γ, this is

$$\begin{aligned}
\tilde{\Pi} &= \frac{1}{1 - \exp(-\Delta)} \int_{y=0}^{\Delta} \int_{x=0}^{\infty} \overline{\gamma}(x + y) g_{N-1}(x) \exp(-y) dx dy \\
&= \frac{1}{1 - \exp(-\Delta)} \int_{x=0}^{\infty} \overline{\gamma}(x) \int_{y=0}^{\min\{x,\Delta\}} g_{N-1}(x - y)\exp(-y) dy dx \\
&\leq \frac{1}{1 - \exp(-\Delta)} \left[\int_{x=\Delta}^{\infty} \overline{\gamma}(x) \int_{y=0}^{x} g_{N-1}(x - y)\exp(-y) dy dx + \exp(-\Delta)^N (1 - \exp(-\Delta))^N \overline{v} \right].
\end{aligned}$$

Now, observe that

$$\begin{aligned}
\int_{y=0}^{x} g_{N-1}(x - y)\exp(-y) dy &= \frac{x^{N-1} - (x - \Delta)^{N-1}}{(N-1)!}\exp(-x) \\
&\leq \frac{\Delta(N-1)x^{N-2}}{(N-1)!}\exp(-x) = \Delta g_{N-1}(x),
\end{aligned}$$

where we have used convexity of x^{N-1}. Thus,

$$\tilde{\Pi} \leq \frac{\Delta}{1-\exp(-\Delta)} \int_{x=0}^{\infty} \overline{\gamma}(x) g_{N-1}(x) dx + \frac{\exp(-\Delta)^N (1-\exp(-\Delta))^N \overline{v}}{1-\exp(-\Delta)}.$$

Since the last two terms converge to zero as $\Delta \to 0$ and $\Delta/(1-\exp(-\Delta)) \to 1$, this implies the claim. □

Finally, since $\tilde{\Pi}$ converges to $\overline{\Pi}$, we can pick Δ sufficiently small so that $\Pi \leq \tilde{\Pi} \leq \overline{\Pi} + \epsilon$. This completes the proof of the proposition. □

Note that every step of the proof of Proposition 5 goes through in the must-sell case, where we replace $\overline{w}$ with $\widehat{w}$, and we skip the step in Claim 7 of proving that the discrete virtual value is non-negative.

B.3 Proof of Theorem 4

Regularity of $\overline{\mathcal{S}}$ is a straightforward implication of Theorem 1 in Milgrom and Weber (1985). For if the mechanism is finite, then since the value function $\overline{w}(s)$ is bounded, the functions $w(s)q_i(m) - t_i(m)$ are equicontinuous across s. Moreover, signals are independent, so that their conditions R1 and R2 is satisfied. Hence, there exists an equilibrium in distributional strategies, which implies existence of an equilibrium in behavioral strategies.

Now, fix a finite information structure $\mathcal{S}$. We will prove existence of a pure-strategy equilibrium in the normal form of $(\overline{\mathcal{M}}, \mathcal{S})$, using Corollary 3.3 of Reny (1999). Let us endow each strategy space $\Delta(\overline{M})^{S_i}$ with the product topology.

Observe that the game is compact and quasiconcave, since payoffs are linear in β. We next verify that the game is payoff secure. We need to show that for any β and $\epsilon > 0$, for every i, there exists a strategy β_i' and an open neighborhood V of β_{-i} such that for all $\beta_{-i}' \in V$,

$$U_i(\beta_i', \beta_{-i}') \geq U_i(\beta) - \epsilon.$$

For some $\Delta > 0$ to be chosen shortly, we define β_i' so that

$$\beta_i'(X|s_i) = \beta_i(X|s_i) + \beta_i(\{0\}|s_i)(\mathbb{I}_{\Delta \in X} - \mathbb{I}_{0 \in X}).$$

In other words, mass on $m_i = 0$ is shifted to Δ. Now, since the strategy β_i' assigns zero probability to $m_i = 0$, the transfer rule is continuous, and the allocation rule is lower semi-continuous on $(0, \infty] \times \overline{M}_{-i}$. We conclude that the payoff

$$\int_{\overline{M}_i} (w(s)q_i(m_i, m_{-i}) - t_i(m_{-i}, m_{-i}))\beta_i'(dm_i|s_i)$$

is lower semi-continuous in m_{-i}. As a result, there exists a neighborhood V of β_{-i} such that for all $\beta'_{-i} \in V$,

$$\begin{aligned}
U_i(\beta'_i, \beta'_{-i}) &= \sum_{s\in S} \int_{\overline{M}} ((w(s) - \underline{v})\overline{q}_i(m) - \overline{t}_i^p(m))\beta'(dm|s)\pi(s) \\
&\geq \sum_{s\in S} \int_{\overline{M}} ((w(s) - \underline{v})\overline{q}_i(m) - \overline{t}_i^p(m))(\beta'_i, \beta_{-i})(dm|s)\pi(s) - \frac{\epsilon}{2} \\
&= U_i(\beta'_i, \beta_{-i}) - \frac{\epsilon}{2}.
\end{aligned}$$

Finally, since $\overline{t}_i^p$ is Lipschitz, we can pick Δ small enough so that

$$\sum_{s\in S} \int_{\overline{M}_{-i}} (\overline{t}_i^p(\Delta, m_{-i}) - \overline{t}_i^p(0, m_{-i}))\beta_{-i}(m_{-i}|s_{-i})\pi(s) \leq \frac{\epsilon}{4}.$$

Moreover, when $\Sigma m_{-i} > 0$, $\overline{q}_i(\Delta, m_{-i}) \geq \overline{q}_i(0, m_{-i})$, and we can always pick Δ small enough so that $(\overline{v} - \underline{v})(\overline{q}_i(\Delta, 0) - 1/N) \leq \epsilon/4$. We therefore conclude that $U_i(\beta'_i, \beta_{-i}) \geq U_i(\beta) - \frac{\epsilon}{2}$. Combining inequalities, we conclude that $U_i(\beta'_i, \beta'_{-i}) \geq U_i(\beta) - \epsilon$ for all $\beta'_{-i} \in V$. This completes the proof of payoff security.

Finally, we verify that the game is reciprocally upper semi-continuous. This is implied by the fact that $\sum_{i=1}^N U_i(\beta)$ is continuous (cf. Reny, 1999, p. 1034), which follows from continuity of the aggregate supply and the transfers. This concludes the proof that $\overline{\mathcal{M}}$ is regular.

C Proofs for the Many-Bidder Limit

To prove Proposition 6, we first need a number of technical results.

Lemma 15. *As N goes to infinity, g_N^C and G_N^C converge pointwise to ϕ and Φ, respectively.*

Proof of Lemma 15. Note that

$$\begin{aligned} g_{N+1}^C(x) &= \sqrt{N} g_{N+1}(\sqrt{N}x + N) \\ &= \sqrt{N}\frac{(\sqrt{N}x + N)^N}{N!} \exp(-\sqrt{N}x - N). \end{aligned}$$

Stirling's Approximation says that

$$0 \le N! - \sqrt{2\pi N}\left(\frac{N}{e}\right)^N \le O\left(\frac{1}{N}\right).$$

Thus, when N is large, $g_{N+1}^C(x)$ is approximately

$$\frac{1}{\sqrt{2\pi}}\left(1 + \frac{1}{\sqrt{N}}\right)^N \exp(-\sqrt{N}x),$$

and hence

$$\log(g_{N+1}^C(x)) \sim \log(1/\sqrt{2\pi}) + N\log\left(1 + \frac{x}{\sqrt{N}}\right) - \sqrt{N}x.$$

Using the mean-value formulation of Taylor's Theorem centered around 0, for every y, there exists a $z \in [0, y]$ such that

$$\log(1 + y) = y - \frac{y^2}{2} + \frac{1}{(1+z)^3}y^3.$$

Plugging in $y = x/\sqrt{N}$, we conclude that

$$\begin{aligned} \log(g_{N+1}^C(x)) &\sim \log(1/\sqrt{2\pi}) + N\frac{x}{\sqrt{N}} - N\frac{1}{2}\left(\frac{x}{\sqrt{N}}\right)^2 + N\frac{1}{(1+z)^3}\left(\frac{x}{\sqrt{N}}\right)^3 - \sqrt{N}x \\ &= \log(1/\sqrt{2\pi}) - \frac{1}{2}x^2 + \frac{1}{(1+z)^3}\frac{x^3}{\sqrt{N}}, \end{aligned}$$

which converges to $\log(1/\sqrt{2\pi}) - \frac{1}{2}x^2$ as N goes to infinity, so $g_{N+1}^C(x)$ converges to $\phi(x) = \exp(-x^2/2)/\sqrt{2\pi}$. Pointwise convergence of G_N to Φ follows from Scheffé's lemma. $\square$

Let us define

$$\tilde{g}(x) = \begin{cases} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) & \text{if } x < 0; \\ \frac{1}{\sqrt{2\pi}}(1 + x)\exp(-x) & \text{otherwise.} \end{cases}$$

Lemma 16. *The function $\tilde{g}(x)|x|$ is integrable, and for all N and x, $|g_N^C(x)| \leq \tilde{g}(x)$.*

Proof of Lemma 16. Note that

$$\int_{x=-\infty}^{\infty} \tilde{g}(x)|x|dx = \int_{x=-\infty}^{0} \phi(x)|x|dx + \frac{1}{\sqrt{2\pi}} \int_{x=0}^{\infty} (1+x)x\exp(-x)dx,$$

which is clearly finite, since the half-normal distribution has finite expectation.

Next, Stirling's Approximation implies that

$$g_{N+1}^C(x) \leq \frac{1}{\sqrt{2\pi}} \left(1 + \frac{x}{\sqrt{N}}\right)^N \exp(-\sqrt{N}x) \equiv \tilde{g}_N(x).$$

Now,

$$\frac{d}{dN} \log(\tilde{g}_N(x)) = \log\left(1 + \frac{x}{\sqrt{N}}\right) + \frac{1}{2}\frac{x}{\sqrt{N}+x} - \frac{x}{2\sqrt{N}},$$

which is clearly zero when $x = 0$, and

$$\begin{aligned}\frac{d}{dx}\frac{d}{dN} \log(\tilde{g}_N(x)) &= \frac{1}{\sqrt{N}+x} - \frac{\sqrt{N}}{2(\sqrt{N}+x)^2} - \frac{1}{2\sqrt{N}} \\ &= \frac{-x^2}{2\sqrt{N}(\sqrt{N}+x)^2},\end{aligned}$$

which is non-positive and strictly negative when $x \neq 0$. As a result, $\tilde{g}_N(x)$ is increasing in N when $x < 0$ and decreasing in N when $x > 0$. Since it converges to $\phi(x)$ in the limit as N goes to infinity, we conclude that for $x < 0$, $g_{N+1}^C(x) \leq \tilde{g}_N(x) \leq \phi(x) = \tilde{g}(x)$, and for $x > 0$, $g_{N+1}^C(x) \leq \tilde{g}_N(x) \leq \tilde{g}_1(x) = \tilde{g}(x)$ as desired. □

Lemma 17. *As N goes to infinity, $\widehat{\gamma}_N^C$ converges almost surely to $\widehat{\gamma}_\infty^C(x) = H^{-1}(\Phi(x))$ and $\widehat{\Gamma}_N^C$ converges pointwise to*

$$\widehat{\Gamma}_\infty^C(x) = \int_{y=-\infty}^{x} \widehat{\gamma}_\infty^C(y)\phi(y)dy.$$

The latter convergence is uniform on any bounded interval.

Proof of Lemma 17. Note that $\widehat{\gamma}_N^C(x) = H^{-1}(G_N^C(x))$. By Lemma 15, $G_N^C(x)$ converges to $\Phi(x)$ pointwise. Thus, if H^{-1} is continuous at $\Phi(x)$, then as N goes to infinity, we must have $\widehat{\gamma}_N^C(x) \to H^{-1}(\Phi(x)) = \widehat{\gamma}_\infty^C(x)$. Since H^{-1} is monotonic, the set of discontinuities has Lebesgue measure zero, so that the pointwise convergence is almost everywhere.

Pointwise convergence of $\widehat{\Gamma}_N^C$ follows from almost sure convergence of $\widehat{\gamma}^C$, combined with the fact that $\widehat{\gamma}_N^C$ is uniformly bounded by $\max\{|\overline{v}|, |c|\}$, so that we can apply the dominated convergence theorem. Moreover, $\widehat{\Gamma}_N^C(x)$ is uniformly Lipschitz continuous across N and x. As a result, the family $\widehat{\Gamma}_N^C(x)$ is uniformly bounded and uniformly equicontinuous. The conclusion about uniform convergence is then a consequence of the Arzela-Ascoli theorem. □

Let us define x^* to be the largest solution to $\widehat{\Gamma}^C_\infty(x^*) = 0$ (which may be $-\infty$). Also, let us define x_N so that $\overline{\Gamma}^C_N$ has a graded interval $[-\sqrt{N-1}, x_N]$. (If there is no graded interval with left end point $-\sqrt{N-1}$, then we let $x_N = -\sqrt{N-1}$.)

Lemma 18. *As N goes to infinity, x_N converges to x^*.*

Proof of Lemma 18. By a change of variables $y = G_N^{-1}(\Phi(x))$, we conclude that

$$\widehat{\Gamma}^C_\infty(x^*) = \int_{x=0}^{x^*} \widehat{\gamma}^C_\infty(x)\phi(x)dx = \int_{x=0}^{G_N^{-1}(\Phi(x^*))} \widehat{\gamma}^C_N(x)g_N(x)dx = \widehat{\Gamma}^C_N(G_N^{-1}(\Phi(x^*))).$$

This integral must be zero by the definition of x^*, so that $x_N \geq G_N^{-1}(\Phi(x^*))$. Since the latter converges to x^* as $N \to \infty$, we conclude $\liminf_{N\to\infty} x_N \geq x^*$.

Next, recall that x_{N+1} solves the equation

$$\begin{aligned}
\widehat{\Gamma}^C_{N+1}(x_{N+1}) &= \widehat{\gamma}^C_{N+1}(x_{N+1}) \int_{x=-\sqrt{N}}^{x_{N+1}} \exp(\sqrt{N}(x - x_{N+1}))g^C_{N+1}(x)dx \\
&= \widehat{\gamma}^C_{N+1}(x_{N+1}) \exp(-\sqrt{N}x_{N+1} - N) \int_{x=-\sqrt{N}}^{x_{N+1}} \exp(\sqrt{N}x + N)g^C_{N+1}(x)dx \\
&= \widehat{\gamma}^C_{N+1}(x_{N+1}) \exp(-\sqrt{N}x_{N+1} - N) \int_{x=-\sqrt{N}}^{x_{N+1}} \sqrt{N}\frac{(\sqrt{N}x + N)^N}{N!}dx \\
&\leq \overline{v} \exp(-\sqrt{N}x_{N+1} - N)\frac{(\sqrt{N}x_{N+1} + N)^{N+1}}{(N+1)!} \\
&\leq \overline{v}\tilde{g}(x_{N+1})\frac{1}{\sqrt{N}},
\end{aligned}$$

where we have used Lemma 16. The last line converges to zero pointwise, so $\widehat{\Gamma}^C_N(x_N)$ must converge to zero as well.

Now, if $z = \limsup_{N\to\infty} x_N > x^*$, then since $\widehat{\Gamma}^C_\infty(z) > \widehat{\Gamma}^C_\infty(x^*) = 0$, we would contradict our earlier finding that $\overline{\Gamma}^C_N(x_N) \to 0$. Thus, $\limsup_{N\to\infty} x_N \leq x^*$, so x_N must converge to x^* as N goes to ∞. □

Lemma 19. *For every $\epsilon > 0$, there exists $\widehat{N}$ such that for all $N > \widehat{N}$, there exists an $x \in [x^* + \epsilon, x^* + 2\epsilon]$ at which $\overline{\gamma}^C_N$ is not graded.*

Proof of Lemma 19. Suppose not. Then there exists a subsequence of N such that for every $x \in [x^* + \epsilon, x^* + 2\epsilon]$, $\overline{\gamma}^C_N(x^* + \epsilon) = \exp(\sqrt{N}(x^* + \epsilon - \tilde{x}))\widehat{\gamma}^C_N(\tilde{x})$ for some $\tilde{x} \geq x^* + 2\epsilon$. Thus, for all $x \leq x^* + \epsilon$, we conclude that

$$\overline{\gamma}^C_N(x) \leq \overline{\gamma}^C_N(x^* + \epsilon) \leq \exp(-\sqrt{N}\epsilon)\overline{v}$$

which converges to zero as N goes to infinity. This implies that $\liminf_{N\to\infty} \overline{\Gamma}^C_N(x^* + \epsilon) = 0$. But $\overline{\Gamma}^C_N(x^* + \epsilon)$ must be weakly larger than $\widehat{\Gamma}^C_N(x^* + \epsilon)$, so

$$0 = \lim\inf_{N\to\infty} \overline{\Gamma}^C_N(x^* + \epsilon) \geq \lim\inf_{N\to\infty} \widehat{\Gamma}^C_N(x^* + \epsilon) = \widehat{\Gamma}^C_\infty(x^* + \epsilon) > 0,$$

a contradiction. □

Lemma 20. *As N goes to infinity, $\overline{\gamma}_N^C$ converges almost surely to*

$$\overline{\gamma}_\infty^C(x) = \begin{cases} 0 \ \textit{if } x < x^*; \\ \widehat{\gamma}_\infty^C(x) \ \textit{if } x \geq x^*. \end{cases}$$

Proof of Lemma 20. Let $x < x^*$. Since $x_N \to x^*$ by Lemma 18, for N sufficiently large, $x_N > (x^* + x)/2$. Since $\overline{\gamma}_N^C(x)$ is graded on $(-\infty, x_N]$, it will be graded at x, and

$$\begin{aligned} \overline{\gamma}_N^C(x) &= \exp(\sqrt{N-1}(x - x_N))\widehat{\gamma}_N^C(x_N) \\ &\leq \exp(\sqrt{N-1}(x - x^*)/2)\overline{v}. \end{aligned}$$

The last line clearly converges to zero pointwise. Since $\overline{\gamma}_N^C(x) \geq 0$ for all N, we conclude that $\overline{\gamma}_N^C(x) \to 0$.

Now consider $x > x^*$ at which $\widehat{\gamma}_\infty^C$ is continuous. Take ϵ so that $x > x^* + 2\epsilon$ and so that $\widehat{\gamma}_\infty^C$ is continuous at $x^* + \epsilon$. Lemma 19 says that there is a $\widehat{N}$ such that for all $N > \widehat{N}$, there exists a point in $[x^* + \epsilon, x^* + 2\epsilon]$ at which the gains function is not graded. Moreover, since $\widehat{\gamma}_N^C(x^* + \epsilon)$ converges to $\widehat{\gamma}_\infty^C(x^* + \epsilon)$, we can pick $\widehat{N}$ large enough and find a constant $\underline{\gamma} > 0$ such that for $N > \widehat{N}$, $\widehat{\gamma}_N^C(x^* + \epsilon) \geq \underline{\gamma}$.

Now, suppose that $\overline{\gamma}_N^C$ is graded at x, with x in a graded interval $[a, b]$. Then $a \geq x^* + \epsilon$, and hence $\widehat{\gamma}_N^C(a) \geq \widehat{\gamma}_N^C(x^* + \epsilon) \geq \underline{\gamma}$. Recall that on $[a, b]$,

$$\overline{\gamma}_N^C(x) = \widehat{\gamma}_N^C(a) \exp(\sqrt{N-1}(x - a)).$$

Since $\widehat{\gamma}_N^C$ is bounded above by $\overline{v}$, it must be that $\widehat{\gamma}_N^C(a) \exp(\sqrt{N-1}(b - a)) \leq \overline{v}$, so

$$\begin{aligned} b - a &\leq \frac{1}{\sqrt{N-1}} \log\left(\frac{\overline{v}}{\widehat{\gamma}_N^C(a)}\right) \\ &\leq \frac{1}{\sqrt{N-1}} \log\left(\frac{\overline{v}}{\underline{\gamma}}\right) \equiv \epsilon_N. \end{aligned}$$

Thus,

$$\widehat{\gamma}_N^C(x - \epsilon_N) \leq \overline{\gamma}_N^C(x) \leq \widehat{\gamma}_N^C(x + \epsilon_N).$$

This was true if $\overline{\gamma}_N^C(x)$ is graded at x, but clearly the inequality is also true if it is not graded at x, in which case $\overline{\gamma}_N^C(x) = \widehat{\gamma}_N^C(x)$. Now, $\widehat{\gamma}_N^C(x) = \widehat{\gamma}_\infty^C(\Phi^{-1}(G_N(x)))$, so

$$\widehat{\gamma}_\infty^C(\Phi^{-1}(G_N^C(x - \epsilon_N))) \leq \overline{\gamma}_N^C(x) \leq \widehat{\gamma}_\infty^C(\Phi^{-1}(G_N^C(x + \epsilon_N))).$$

As $N \to \infty$, the left and right hand sides converge to $\widehat{\gamma}_\infty^C(x)$ from the left and right, respectively. Since $\widehat{\gamma}_\infty^C$ is continuous at x, we conclude that $\overline{\gamma}_N^C(x) \to \widehat{\gamma}_\infty^C(x)$. The lemma follows from the fact that the monotonic function $\widehat{\gamma}_\infty^C$ is continuous almost everywhere.

$\square$

Proof of Proposition 6. We will argue that

$$Z_{N+1} = \sqrt{N} \int_{x=0}^{\infty} \overline{\gamma}_{N+1}(x)(g_{N+1}(x) - g_N(x))dx$$

converges to a positive constant as N goes to infinity. Since this is $\sqrt{N}$ times the difference between ex ante gains from trade and profit, this will prove the result.

To that end, observe that

$$Z_{N+1} = \sqrt{N} \int_{x=0}^{N/2} \overline{\gamma}_{N+1}(x)(g_{N+1}(x) - g_N(x))dx + \int_{x=-\sqrt{N}/2}^{\infty} \overline{\gamma}^C_{N+1}(x) g^C_{N+1}(x) \frac{Nx}{\sqrt{N}x + N} dx.$$

We claim that the first integral converges to zero as $N \to \infty$. Note that $g_{N+1}(x) \le g_N(x)$ if and only if $x \le N$. Therefore,

$$\begin{aligned}
\left| \sqrt{N} \int_{x=0}^{N/2} \overline{\gamma}_{N+1}(x)(g_{N+1}(x) - g_N(x))dx \right| &\le (\overline{v} + c)\sqrt{N} \int_{x=0}^{N/2} (g_N(x) - g_{N+1}(x))dx \\
&= (\overline{v} + c)\sqrt{N}(G_N(N/2) - G_{N+1}(N/2)) \\
&= (\overline{v} + c)\sqrt{N} g_{N+1}(N/2) \\
&= (\overline{v} + c)\sqrt{N} \frac{(N/2)^N \exp(-N/2)}{N!} \\
&\sim (\overline{v} + c)\sqrt{N} \frac{(N/2)^N \exp(-N/2)}{\sqrt{2\pi N}(N/e)^N} \\
&= (\overline{v} + c) \frac{1}{\sqrt{2\pi}} \exp(-N(\log(2) - 1/2)),
\end{aligned}$$

where we have again used Stirling's approximation between the third-to-last and second-to-last lines. The last line converges to zero as N goes to infinity.

Now consider the second integral in the formula for Z_{N+1}. By Lemma 16, the integrand is bounded above in absolute value by the integrable function $\overline{v}\tilde{g}(x)|x|$. Moreover, from Lemmas 15 and 20, we know that the integrand converges pointwise to $\overline{\gamma}^C_{\infty}(x)\phi(x)x$. The dominated convergence theorem then implies that as N goes to infinity, Z_N converges to

$$\int_{x=-\infty}^{\infty} \overline{\gamma}^C_{\infty}(x)\phi(x)x dx,$$

which is strictly positive because $\overline{\gamma}^C_{\infty}$ is strictly increasing.

The proof goes through for the must-sell guarantee, if we replace $\overline{\gamma}^C_N$ with $\widehat{\gamma}^C_N$. $\square$

To prove Proposition 8, we need a few more intermediate results. Let $\overline{G}_N(x) \equiv G_N(Nx)$ be the cumulative distribution for the mean of N independent standard exponential random variables. Define $\overline{F}_N(x) \equiv \exp(N(1 - x + \log(x)))$. Clearly, $\overline{F}_N(x)$ is a cumulative distribution for $x \in [0, 1]$, $\overline{F}_N(0) = 0$ and $\overline{F}_N(1) = 1$. Finally, define the function $D_N(\alpha)$:

$$D_N(\alpha) \equiv \begin{cases} \frac{1}{\overline{F}_N^{-1}(\alpha)} & \alpha \in [0, 0.4], \\ 1.1 & \alpha \in (0.4, 1]. \end{cases}$$

The choices of 0.4 and 1.1 in $D_N(\alpha)$ are arbitrary: any number less than 1/2 and more than 1, respectively, will work.

Lemma 21. *When $\widehat{N}$ is sufficiently large, we have $\overline{\mu}_N(G_N^{-1}(\alpha)) \le D_{\widehat{N}}(\alpha)$ for all $N \ge \widehat{N}$ and $\alpha \in [0,1]$.*

Proof of Lemma 21. We first apply the theory of large deviations to the exponential distribution. Let $\Lambda(t)$ be the logarithmic moment generating function for the exponential distribution:

$$\Lambda(t) = \log\left(\int_{x=0}^{\infty} \exp(xt - x)\,dx\right) = \begin{cases} \infty & t \ge 1, \\ -\log(1-t) & t < 1. \end{cases}$$

Let $\Lambda^*(x)$ be the Legendre transform of $\Lambda(t)$:

$$\Lambda^*(x) = \sup_{t\in\mathbb{R}}\{xt - \Lambda(t)\} = \begin{cases} \infty & x \le 0, \\ x - 1 - \log x & x > 0. \end{cases}$$

Cramér's theorem (or the Chernoff bound; see Theorem 1.3.12 in Stroock, 2011) then states that for any N,

$$\overline{G}_N(x) \le \exp(-N\Lambda^*(x)) = \overline{F}_N(x)$$

for every $x \in [0,1]$; or equivalently, $\overline{F}_N^{-1}(\alpha) \le \overline{G}_N^{-1}(\alpha)$ for every $\alpha \in [0, \overline{G}_N(1)]$.

By the law of large numbers, when $\widehat{N}$ is sufficiently large, we have $\overline{G}_N(1) \ge 0.4$ and $1/\overline{G}_N^{-1}(0.4) \le 1.1$ and for all $N \ge \widehat{N}$. The claim of the lemma then follows from two cases:

If $\alpha \in [0, 0.4]$, then we have

$$\overline{\mu}_N(G_N^{-1}(\alpha)) \le \frac{1}{\overline{G}_N^{-1}(\alpha)} \le \frac{1}{\overline{F}_N^{-1}(\alpha)} \le \frac{1}{\overline{F}_{\widehat{N}}^{-1}(\alpha)} = D_N(\alpha)$$

since $\overline{G}_N(1) \le 0.4$ when $N \ge \widehat{N}$, and $\overline{F}_N(x) \le \overline{F}_{\widehat{N}}(x)$ for all $N \ge \widehat{N}$ and $x \in [0,1]$.

If $\alpha \in (0.4, 1]$, then

$$\overline{\mu}_N(G_N^{-1}(\alpha)) \le \frac{1}{\overline{G}_N^{-1}(\alpha)} \le \frac{1}{\overline{G}_N^{-1}(0.4)} \le 1.1 = D_N(\alpha)$$

since $\overline{G}_N^{-1}(\alpha)$ is increasing in α, and $1/\overline{G}_N^{-1}(0.4) \le 1.1$ when $N \ge \widehat{N}$. $\square$

Lemma 22. *When N is sufficiently large, we have*

$$\int_{\alpha=0}^{1} D_N(\alpha)\,dH^{-1}(\alpha) < \infty.$$

Proof of Lemma 22. Since $G_N(x) = 1 - \sum_{k=1}^N g_k(x)$, we have:

$$\begin{aligned}
\overline{G}_N(x) &= 1 - \sum_{k=1}^{N} \exp(-Nx)\frac{(Nx)^{k-1}}{(k-1)!} \\
&= 1 - \exp(-Nx)\left(\exp(Nx) - \sum_{k=N}^{\infty}\frac{(Nx)^k}{k!}\right) \geq \exp(-Nx)\frac{(Nx)^N}{N!}.
\end{aligned}$$

Clearly there exists a $\overline{x} \in (0,1)$ such that

$$\overline{F}_{N+1}(x) = \exp((N+1)(1-x))x^{N+1} \leq \exp(-Nx)\frac{(Nx)^N}{N!} \leq \overline{G}_N(x)$$

for all $x \in [0, \overline{x}]$. We therefore have $D_{N+1}(\alpha) = 1/\overline{F}_{N+1}^{-1}(\alpha) \leq 1/\overline{G}_N^{-1}(\alpha)$ for all $\alpha \in [0, \overline{\alpha}]$, where $\overline{\alpha} = \min(\overline{F}_{N+1}(\overline{x}), 0.4)$. As a result,

$$\int_{\alpha=0}^{1} D_{N+1}(\alpha)\, dH^{-1}(\alpha) \leq \int_{\alpha=0}^{\overline{\alpha}} \frac{1}{\overline{G}_N^{-1}(\alpha)}\, dH^{-1}(\alpha) + \int_{\alpha=\overline{\alpha}}^{1} \max\left(\frac{1}{\overline{F}_{N+1}^{-1}(\overline{\alpha})}, 1.1\right) dH^{-1}(\alpha) < \infty$$

holds whenever we have

$$\int_{\alpha=0}^{1} \frac{1}{\overline{G}_N^{-1}(\alpha)}\, dH^{-1}(\alpha) = \int_{x=0}^{\infty} \frac{N}{x}\, d\widehat{w}_N(x) < \infty.$$

But finiteness of the last integral follows from the left-tail condition. $\square$

Lemma 23. *Suppose* $\lim_{N\to\infty} y_N \in (0, \infty)$. *Then* $\lim_{N\to\infty} \overline{\mu}_{N+1}(\sqrt{N}y_N + N) = 1$.

Proof of Lemma 23. We first argue that for almost every y, $\overline{\mu}_{N+1}(\sqrt{N}y + N)$ tends to 1 as $N \to \infty$. For this we recall x^* and x_N from Lemmas 18–20.

Consider first $y < x^*$. For N sufficiently large, the gains function is graded at y, and hence

$$\overline{\mu}_{N+1}(\sqrt{N}y + N) = C(0, \sqrt{N}x_{N+1} + N) = \frac{N+1}{\sqrt{N}x_{N+1} + N}.$$

Since we have already shown that $x_N \to x^*$ (Lemma 18), we conclude that $\overline{\mu}_{N+1}(\sqrt{N}y+N)$ goes to 1.

Now consider $y > x^*$ at which $\widehat{\gamma}_\infty^C$ is continuous. If the gains function is not graded at y, then $\overline{\mu}_{N+1}(\sqrt{N}y + N) = N/(\sqrt{N}y + N)$. If the gains function is graded at y, then the length of the graded interval $[a, b] \ni y$ in CLT units is less than $\epsilon_N = \overline{v}/(\overline{\gamma}\sqrt{N}$ for some $\overline{\gamma} > 0$ independent of N (see Lemma 20). Therefore, we have

$$\frac{N}{\sqrt{N}(y + \epsilon_N) + N} \leq \overline{\mu}_{N+1}(\sqrt{N}y + N) \leq \frac{N}{\sqrt{N}(y - \epsilon_N) + N},$$

since $\lim_{z\nearrow a}\overline{\mu}_{N+1}(\sqrt{N}z+N)=\frac{N}{\sqrt{N}a+N}$ and $\lim_{z\searrow b}\overline{\mu}_{N+1}(\sqrt{N}z+N)=\frac{N}{\sqrt{N}b+N}$. As a result, $\overline{\mu}_{N+1}(\sqrt{N}y+N)$ is squeezed to 1 as N goes to infinity.

We conclude that $\overline{\mu}_{N+1}(\sqrt{N}y+N)$ goes to 1 for $y>x^*$ at which $\widehat{\gamma}^C_\infty$ is continuous. Since $\widehat{\gamma}^C_\infty(y)$ is a monotone function of y, it is continuous at almost every y, so the convergence $\overline{\mu}_N\to 1$ is almost everywhere.

Finally, suppose $\lim_{N\to\infty}y_N=y\in(0,\infty)$. Choose y' and y'' such that $y\in(y',y'')$ and such that

$$\lim_{N\to\infty}\overline{\mu}_{N+1}(\sqrt{N}y'+N)=1=\lim_{N\to\infty}\overline{\mu}_{N+1}(\sqrt{N}y''+N).$$

When N is sufficiently large, we have $y_N\in(y',y'')$, so

$$\overline{\mu}_{N+1}(\sqrt{N}y''+N)\le\overline{\mu}_{N+1}(\sqrt{N}y_N+N)\le\overline{\mu}_{N+1}(\sqrt{N}y'+N).$$

Taking the limit as $N\to\infty$, we conclude $\lim_{N\to\infty}\overline{\mu}_{N+1}(\sqrt{N}y_N+N)=1$. □

Proof of Proposition 8. We first prove that

$$\lim_{N\to\infty}\overline{\lambda}_N(v;H)\to v-c \tag{35}$$

for every $v\in[\underline{v},\overline{v}]$.

Replacing $\overline{\mu}_N$ by 1 in equation (19), the definition of $\overline{\lambda}_N(v;H)$, we have

$$\begin{aligned}\overline{\Pi}_N(H)+\int_{y=0}^{\infty}G_N(y)\,d\widehat{w}_N(y)-\int_{\nu=v}^{\overline{v}}d\nu=&\overline{\Pi}_N(H)+\left(\overline{v}-\int_{y=0}^{\infty}g_N(y)\,\widehat{w}_N(y)\right)-(\overline{v}-v)\\=&\overline{\Pi}_N(H)-\int v\,dH(v)+v.\end{aligned}$$

Since $\lim_{N\to\infty}\overline{\Pi}_N(H)\to\int_V v\,dH(v)-c$, to prove (35), it suffices to prove that

$$\lim_{N\to\infty}\int_{y=0}^{\infty}|1-\overline{\mu}_N(y)|\,d\widehat{w}_N(y)=0.$$

Changing variables, we can rewrite the above equation as:

$$\lim_{N\to\infty}\int_{\alpha=0}^{1}|1-\overline{\mu}_N(G_N^{-1}(\alpha))|\,dH^{-1}(\alpha)=0. \tag{36}$$

We note that Stieltjes integration with respect to $dH^{-1}(\alpha)$ is equivalent to a Lebesgue integration with respect to the finite measure ω on $[0,1]$ satisfying $\omega([s,t))=H^{-1}(t)-H^{-1}(s)$, $0\le s\le t\le 1$, and $\omega(\{1\})=0$. The left-tail condition (1) implies that

$$\omega(\{0\})=\lim_{\alpha\to0}\omega([0,\alpha))=\lim_{\alpha\to0}H^{-1}(\alpha)-H^{-1}(0)\le\lim_{\alpha\to0}C\cdot G_N^{-1}(\alpha)^{\varphi}=0$$

for some $\varphi>1$ and $C>0$. Therefore, $\omega(\{0,1\})=0$.

The central limit theorem implies that $\lim_{N\to\infty}(G_N^{-1}(\alpha)-(N-1))/\sqrt{N-1}=\Phi^{-1}(\alpha)$ for every $\alpha \in (0,1)$. Therefore, Lemma 23 implies $\lim_{N\to\infty}\overline{\mu}_N(G_N^{-1}(\alpha)) = 1$ for every $\alpha \in (0,1)$. Moreover, Lemmas 21 and 22 imply that there exists a $\widehat{N}$ such that for all $N \geq \widehat{N}$, the integrand $|1-\overline{\mu}_N(G_N^{-1}(\alpha))|$ in (36) is dominated by $1+D_{\widehat{N}}(\alpha)$ which is integrable with respect to ω. Therefore, equation (36) follows from the dominated convergence theorem, from which equation (35) follows.

Finally, using the definition of $\overline{\lambda}_N(v;H)$, we have

$$\overline{\lambda}_N(v;H) \leq \overline{\Pi}_N(H)+\int_{y=0}^{\infty}\overline{\mu}_N(y)(1+G_N(y))\,d\widehat{w}_N(y) \leq (\overline{v}-c)+2\int_{\alpha=0}^{1}D_{\widehat{N}}(\alpha)\,dH^{-1}(\alpha) < \infty,$$

for all $v \in [\underline{v},\overline{v}]$ and $N \geq \widehat{N}$, where the inequalities follow from Lemmas 21 and 22. Thus

$$\lim_{N\to\infty}\int_V \overline{\lambda}_N(v;H)\,dH'(v) = \int_V v\,dH'(v) - c$$

follows the dominated convergence theorem using (35).

The proof for the must-sell $\widehat{\lambda}_N(v;H)$ is identical, after replacing $\overline{\mu}_N(x)$ with $\widehat{\mu}_N(x) = (N-1)/x$ and $\overline{\Pi}_N(H)$ with $\widehat{\Pi}_N(H)$. $\square$