Appendix D
Properties of Uniform Price Auctions for IPOs

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While there is a considerable body of literature investigating properties of various auction models, there are issues that can only be adequately analyzed by a quantitative model. We do this by means of a calibration exercise below. In addition, we investigate the properties of some off-equilibrium allocations, such as games with naïve participants.

Description of the model IPO auction

There are K identical units of good (e.g. equal-sized lots of IPO shares) that are being auctioned to N risk-neutral bidders. Each unit has the same value V to every bidder. V is unknown in advance and is distributed according to a distribution function $G(V)$. Every bidder i receives a signal S_i about the true value of V, such that $E\{ S_i \}=V$.

Upon receiving their signals S_i, the bidders determine their bids $B_i(S_i)$. The auctioneer collects the bids, determines the clearing price (equal to $(K+1)$'st highest bid in case of a uniform-price auction) and allocates one (and only one) lot of shares to every bidder whose bid is above the clearing price.

In the analysis below, we shall assume that both V and S_i are distributed lognormally[1], and the S_i's are independent of each other. In particular, we consider a small IPO with an ex ante expected value of $15MM, with bidders competing for 15 equal round lots of 100,000 shares each, so that the expected value of each share is $10. The value V per share is distributed lognormally with E(V) equal to 10 and standard deviation of 0.30 for log(V) -- corresponding to a standard deviation of 30% for the continuously compounded rate of return to an uninformed investor in the stock. It is also possible to obtain additional private signal, S, about the security's value, centered at the actual value and also with a standard deviation of 30% for log(S) conditional on the realized value, V of a share.

An *equilibrium allocation* is such where for each bidder i his strategy B_i is the optimal response to the collection of other bidders' strategies $\{ B_j \}_{j\neq i}$:

$$B_i(S_i) \in \operatorname{argmax} E_{V,\{S_j\}_{j\neq i}} \{(V-p)W_i\}$$

Where $p\left(V,\{S_j\}_{j\neq i},\{B_j(\cdot)\}_{j\neq i}\right)$ is the clearing price and $W_i\left(B_i,V,\{S_j\}_{j\neq i},\{B_j(\cdot)\}_{j\neq i}\right)$ is an indicator variable, equal to 1 if $B_i(S_i)>p$ and 0 if $B_i(S_i)<p$. Ties are assumed to be broken at random, so that W_i=(Number of bidders bidding B_i)/N if $B_i(S_i)=p$. Note that if the unconditional signal distribution has no mass points and all bidders' strategies are strictly increasing, a tie is a probability zero event.

A *symmetric allocation* is such where $B_i=B$, i.e. when all bidders' strategies are the same.

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[1] ln V and ln S_I both Gaussian, with $EV = \mu$, $E\{S_i \mid V\} = V$

The game described above has a unique symmetric equilibrium, described first in Milgrom (1981), and which is given by $B_i(s) = \mathrm{E}\{V \mid S_i=s,\ S_{-i}^{(K)}=s\}$, where $S_{-i}^{(K)}$ is the K^{th} highest signal of all agents other than *I*, and the expectation is taken over the joint distribution of V and $\{S_{-i}\}$.

Bidders' strategy

Figure 1 gives the bidding functions of several different types of bidders. The 45^o line (bold dashed line in the figure) corresponds to the strategy of a naïve bidder who always bids his own signal – a particularly simple and easy to implement rule of thumb. The bold solid line corresponds to that of a somewhat less naïve bidder who bids his posterior estimate of the value, conditional on his own signal only. His bid lies between the prior mean of *10* and the value of his signal.

Figure 2 shows expected profit of such bidders in symmetric allocations. Note that these are not in equilibrium, as deviations can in principle be profitable. Such allocations can, however, arise when bidders are either not fully optimizing or have not yet learned enough about the game to determine optimal play. In particular, one can see that bidders' profits remain positive as long as their numbers are low (less than 34 for "bid your own signal" bidders, and 31 for "bid your posterior estimate" bidders). Some naïve bidders may consider such profits as a vindication of their strategy and fail to update. However, such naïve bidders would take substantial losses should more people decide to participate. The remaining three lines in Figure 1 correspond to bids in the unique symmetric equilibrium described in Milgrom (1981) for varying number of participants. Note in particular how the equilibrium bids are practically indistinguishable from naïve "bid your own signal" strategy when the number of participants *N* is close to double the number of lots *K*. Correct determination of the equilibrium strategy, however, requires substantial sophistication and knowledge about the model and its parameters (such as the number of participants) on part of every bidder. As we shall see below, even small departures from these assumptions have a substantial effect on equilibrium allocations.

The solid line in Figure 2 shows equilibrium expected profits to bidders in the symmetric equilibrium with $N=30$. Unlike the allocations with naïve participants, these profits are always positive[2]. They are also quite low. Such low profits can generally be discouraging to participants, and in presence of modest costs to participation and learning about the model and the true value they would limit the number of bidders that would be willing to join the auction. In addition, such costs may encourage free-riding on the efforts of others as we shall see below.

Figure 3 shows the expected underpricing per share (equal to the negative of the profit of a winning bidder). As follows from the previous discussion, in equilibrium one can always expect a modest underpricing, but with "rule of thumb" bidders one can potentially see both very overpriced and very underpriced issues.

[2] Otherwise sitting out the auction by submitting a bid of zero would have been the dominant strategy

Figure 1.1_1 Explains how the bid will vary with information (signal) for the following cases:

a. Naively bidding own signal

b. More sophisticated: Bid the expected value given signal – not as easy computationally

c. Bid taking winner's curse into account (Milgrom solution) – even more sophistication – and all bidders must be equally sophisticated and all must know the information structure and how others are bidding

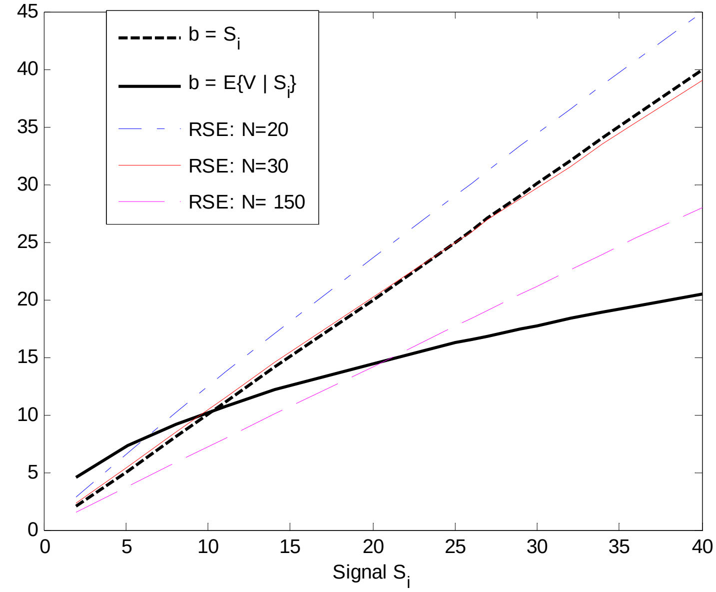


Figure 2: 1_2 Examines how much E(profit) each type of bidding will give rise to under symmetric behavior

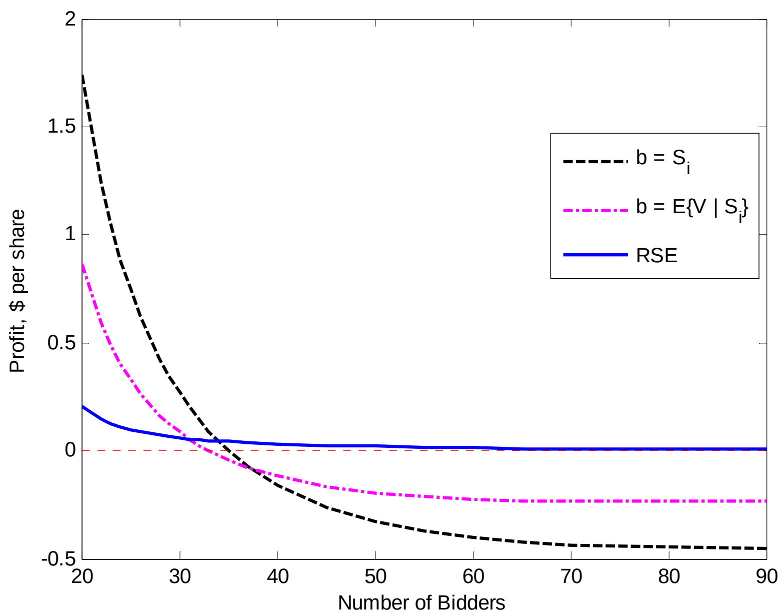
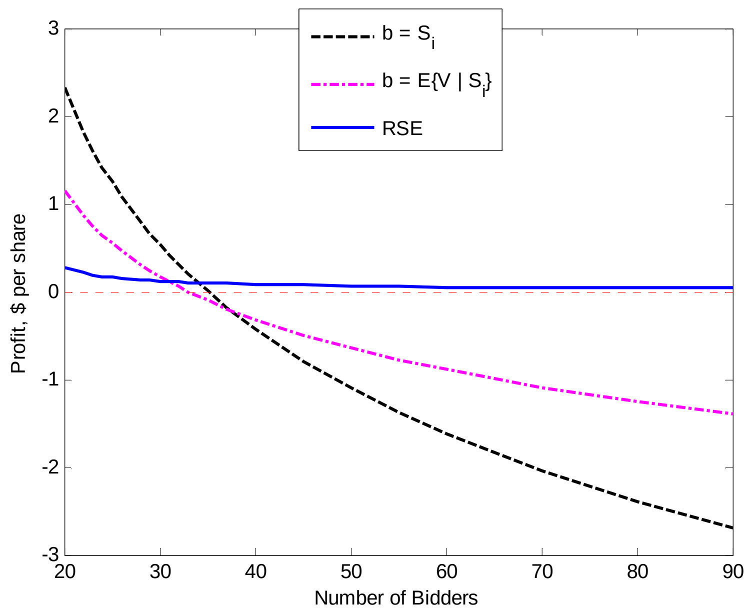


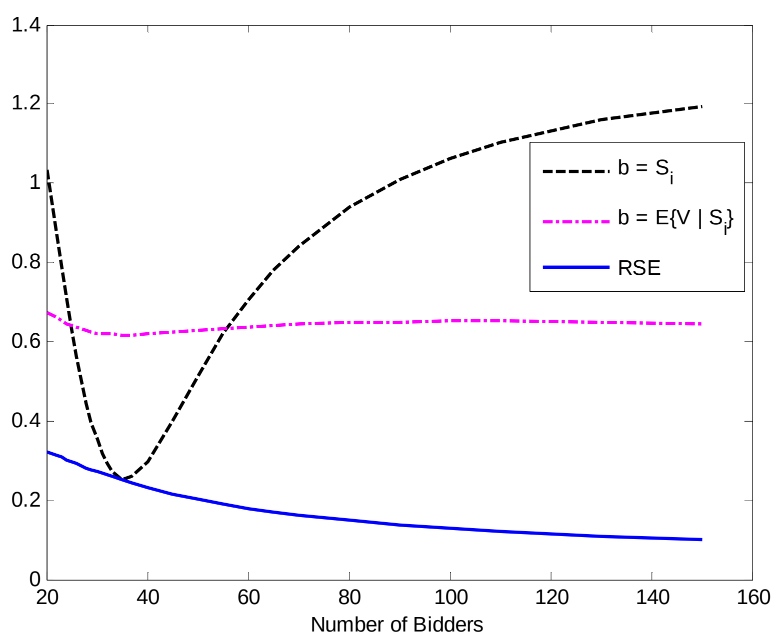
Figure 3: 1_3 E(Profit conditional on Winning) – same as expected underpricing



As seen from

Figure 4, the risk to participants is also higher in "rule-of-thumb" allocations compared to the symmetric equilibrium, except for the bidders who bid their own signal when *N=2K*.

Figure 4: 1_4: Standard deviation of the profits (underpricing) conditional on winning



Effects of uncertainty

One underlying source of aggregate uncertainty is the randomness in *V*, which can only be partially revealed in any mechanism as long as the number of signals generated is finite. Figure 5 shows the auction discount as a function of the standard deviation of V, when the log signals' standard deviation is kept constant at

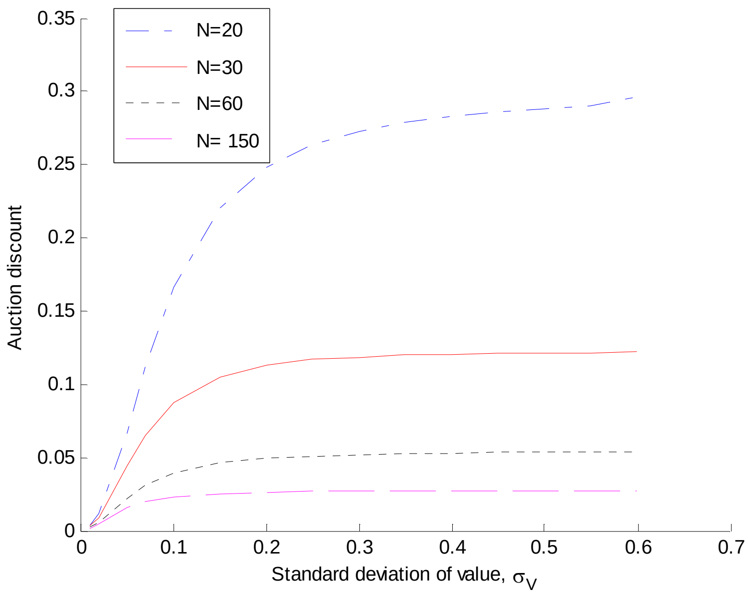


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Figure 6 shows the standard deviation of the discount.

Figure 5: 5_1: Effects of uncertainty – V is more uncertain – signal has same precision; expected discount

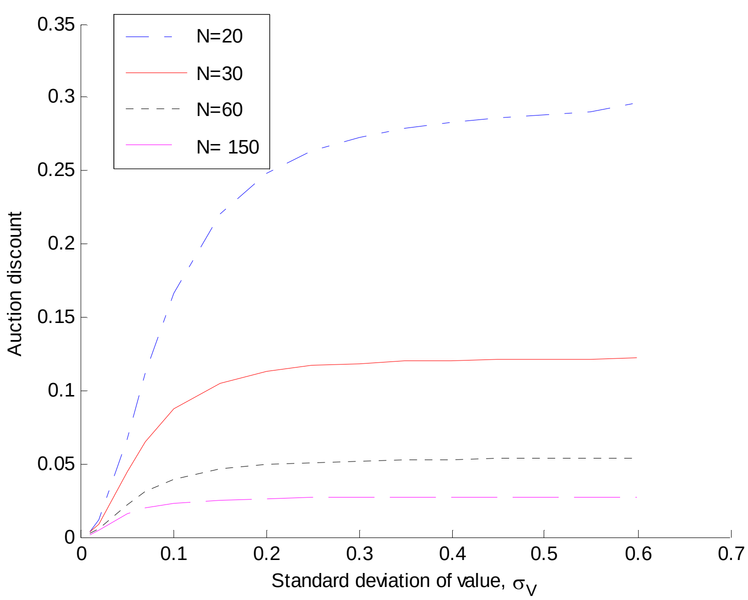


Figure 6: 5_3 : Effects of uncertainty – V is more uncertain – signal has same precision; std dev of discount (profit conditional on winning)

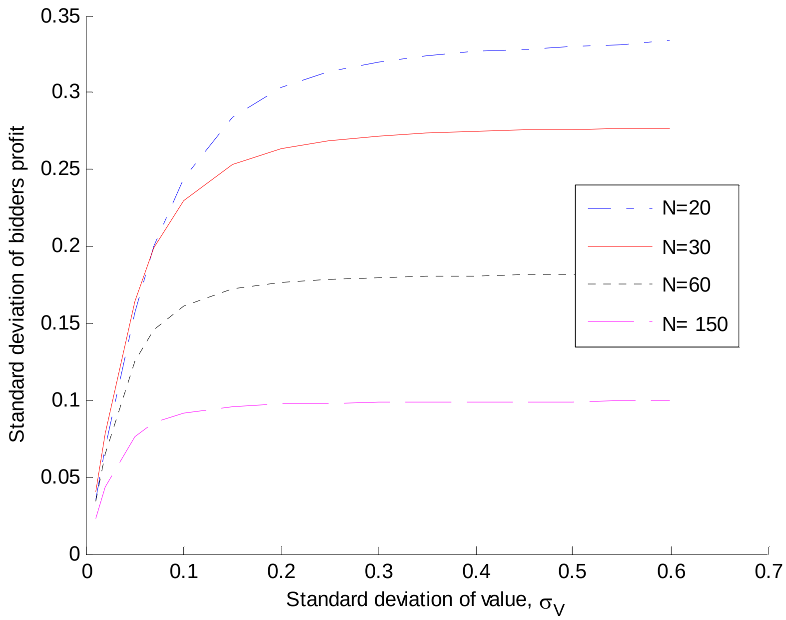


Figure 7: 4_1: Effects of uncertainty – V uncertainty same – signal precision varies – expected discount

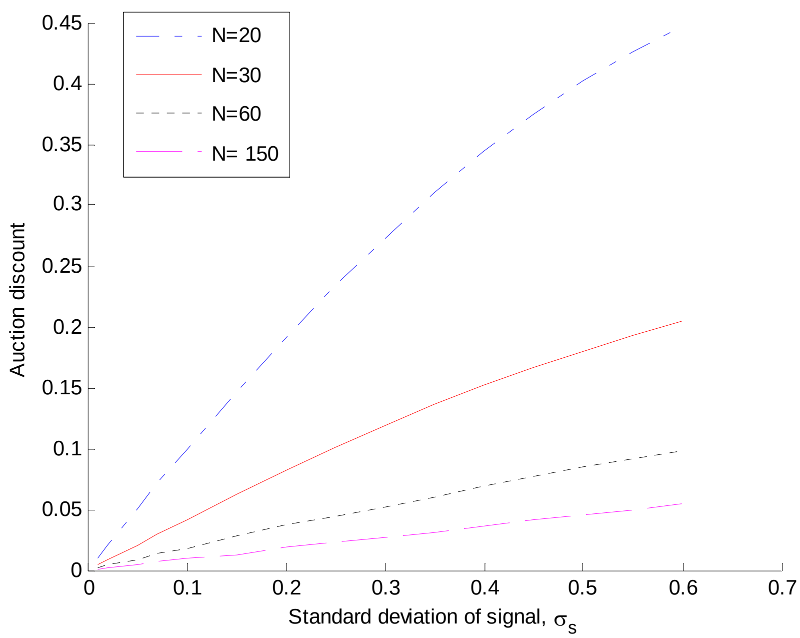


Figure 8: 4_3 : Effects of uncertainty – V uncertainty same – signal precision varies – std dev of discount (profit conditional on winning)

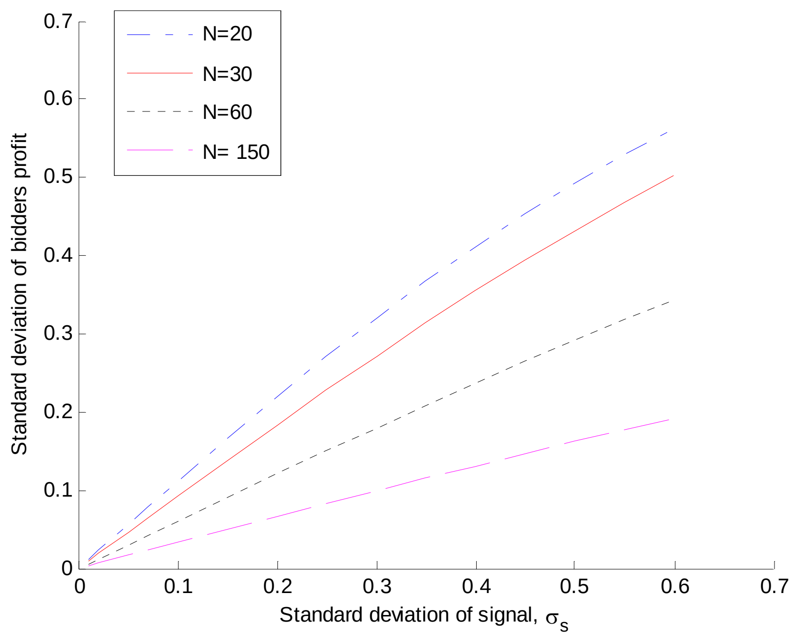
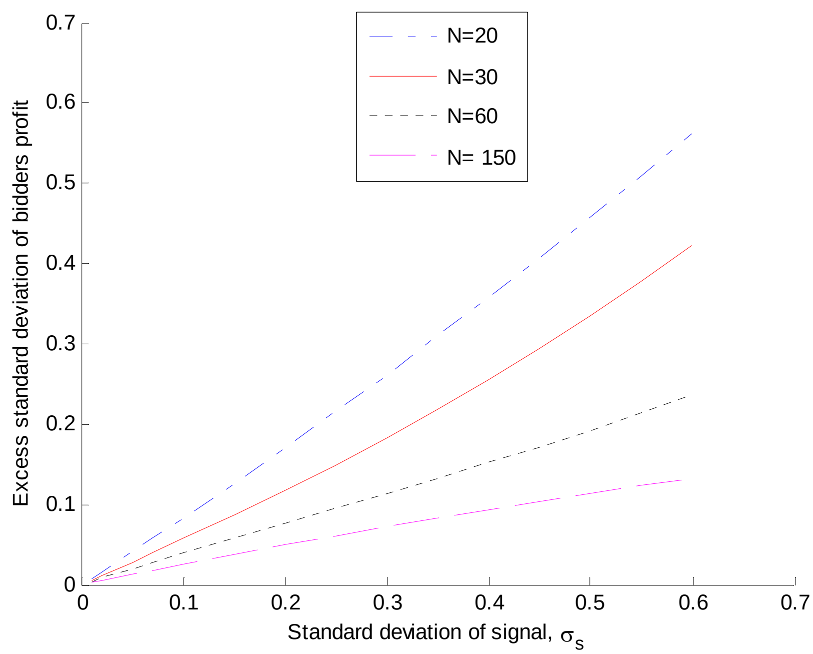


Figure 9. Excess uncertainty (standard deviation of the difference between mean signal and clearing price)



Effects of risk-aversion

Under risk aversion, the symmetric equilibrium strategy is given by

$$B_i(s) = \text{argmax } \mathrm{E}\{u(V-B_i)\geq 0 \mid S_i=s,\ S_{-i}^{(K)}=s\}$$

For the purposes of this exercise, we will consider constant relative risk aversion utility $u(c)=c^{1-A}/(1-A)$, with initial capital of $30MM for each bidder. From Figure 10 we can see that more risk-averse bidders demand higher expected profit, although even for very high values of A profit is modest (even when $N=20$ and $A=55$ the expected profit is only $0.6, or 6% of ex ante expected share value of $10) and quickly declining with an increase in number of bidders. Figure 11 shows that underpricing follows a similar pattern.

Figure 10: Effects of risk aversion on bidders expected profit

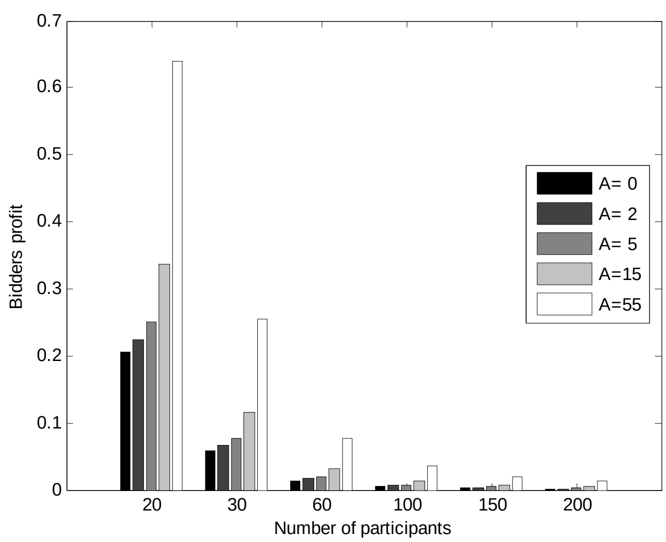
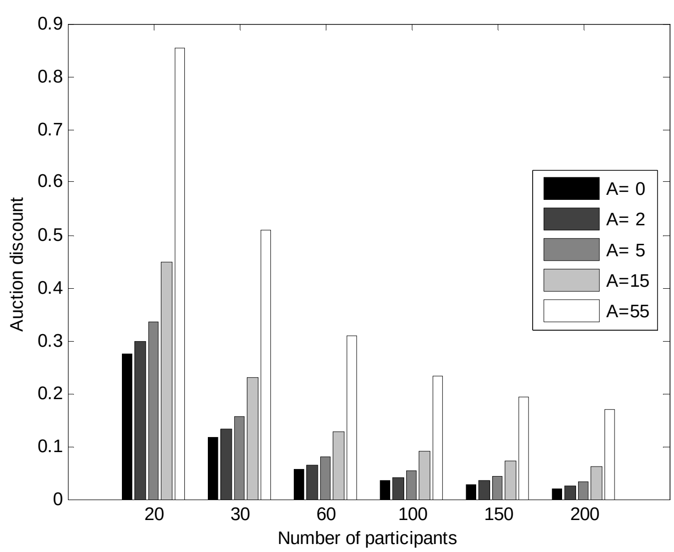
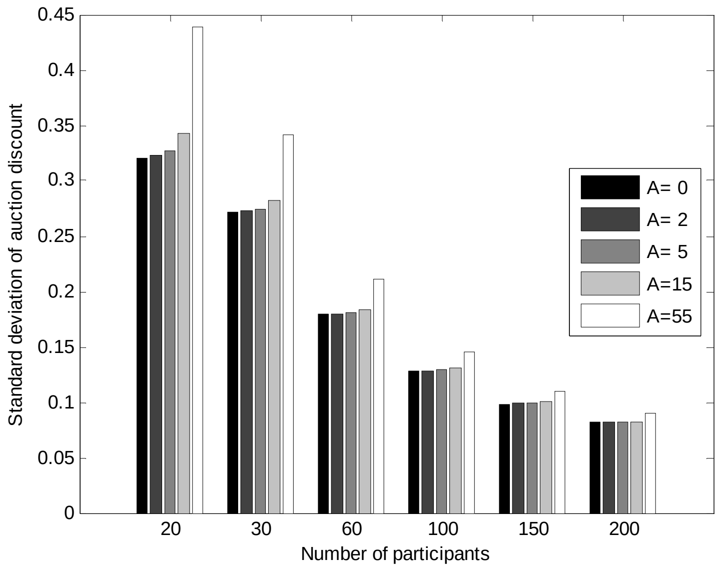


Figure 11: Effects of risk aversion on underpricing



From Figure 12 we can see that price discovery in auctions with more risk-averse participants is slightly less efficient, although in all cases uncertainty about the discount quickly declines with *N*. As we shall see below, this follows from the assumption that the bidders know precisely the number of participants and can condition their strategies on that knowledge. If the number of participants is uncertain, this will no longer be true, and an increase in expected *N* may cause an increase in both discount and risk.

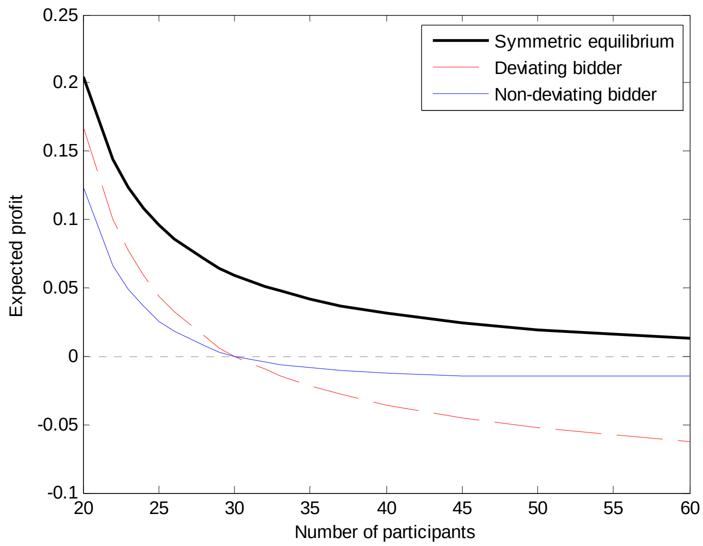
Figure 12: Effects of risk aversion on standard deviation of auction discount



Deviating bids and free-riding

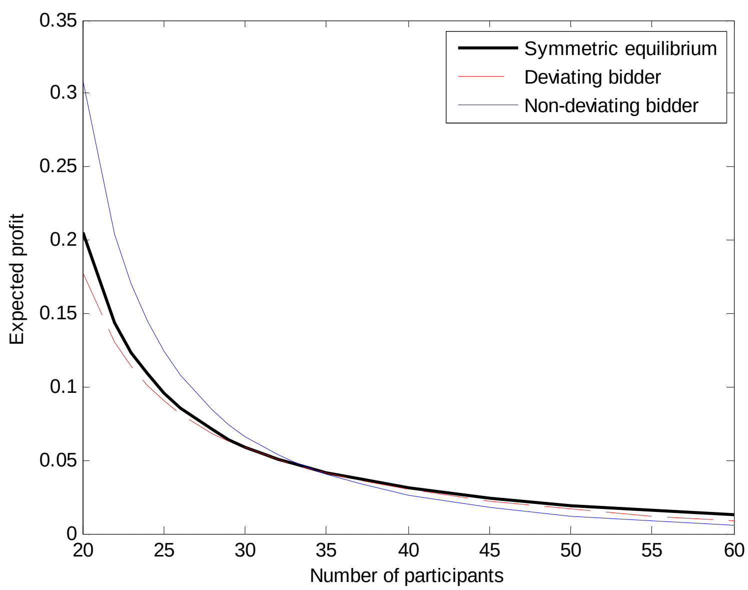
Now suppose that unbeknownst to others, one of the agents instead of following the equilibrium strategy says that he is willing to buy one unit at whatever was the auction clearing price (so called "market bid"), effectively bidding an infinitely high amount. The dashed and solid thin line in Figure 15 show correspondingly his expected profit and other bidders' expected profit, depending on the number of participants.

Figure 13: 2_1: Effect of Free Rider -- one bidder makes a high bid – others do not know that one person is doing that



Note that just one such deviating bidder out of 80 can make everyone's profits turn into losses. Other types of deviations also affect equilibrium outcome, although not necessarily quite as much – see Figure 14 as an example of a situation where one bidder in the symmetric equilibrium chooses to just bid his own signal as opposed to the equilibrium strategy.

Figure 14: 8_1: Effect of mistake; one bidder bids own signal – others do not know that one person is doint that



Such deviations, of course, cannot be profitable in equilibrium. However, there are factors such as information gathering costs that can change this result, and consequently affect the equilibrium outcome.

Information gathering

To put these numbers into perspective, suppose a participating institution would have to incur a cost of $10,000 to acquire the private signal, or 1% of the amount bid. This corresponds, for example, to two weeks of labor of a $250,000 a year analyst.

If the underwriter is able to successfully screen participants and only allow those who actually performed due diligence and did similar analysis, such expenditure would be profitable when the number of participants does not exceed 24, which would be the equilibrium number of bidders in this situation.

Now suppose that the deviating bidder, does not expend the effort necessary to obtain additional information about the security. Instead, he participates in the auction with an arbitrary high bid. Figure 15 shows profits (net of information acquisition) to the agents in this case – as we can see, the deviating agent is now better off than both the non-deviating agents and the agents in the symmetric equilibrium. However, other bidders are much worse off now. Risk to all agents also goes up substantially, as seen from Figure 16.

Figure 15: 2_2: Effect of Free Rider one bidder makes a high bid – others do not know that one person is doing that; information is costly – almost the same as Fig 10.

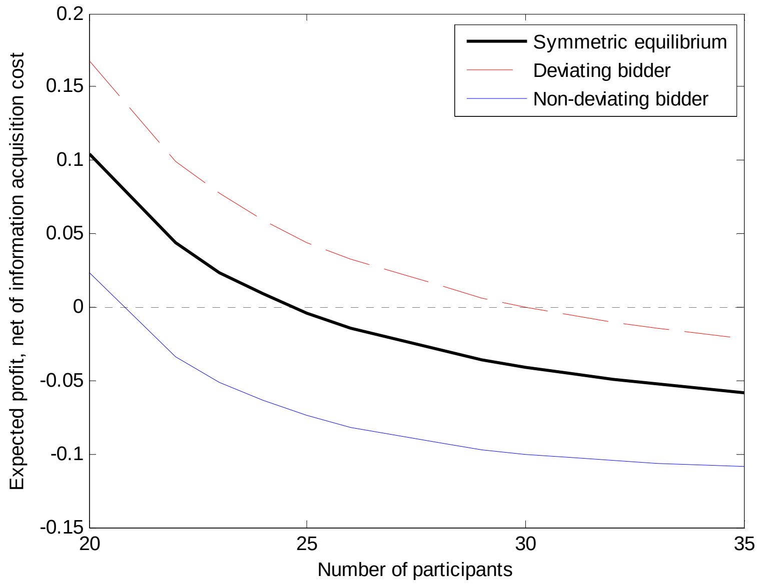
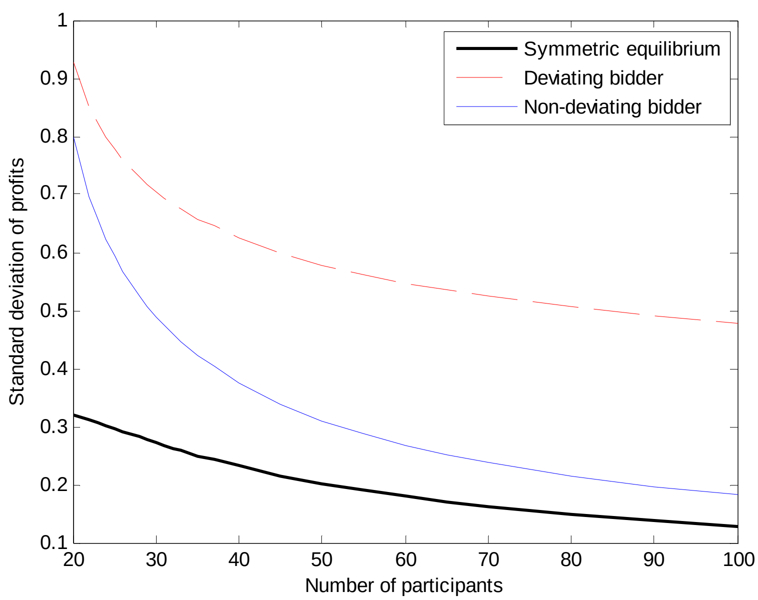
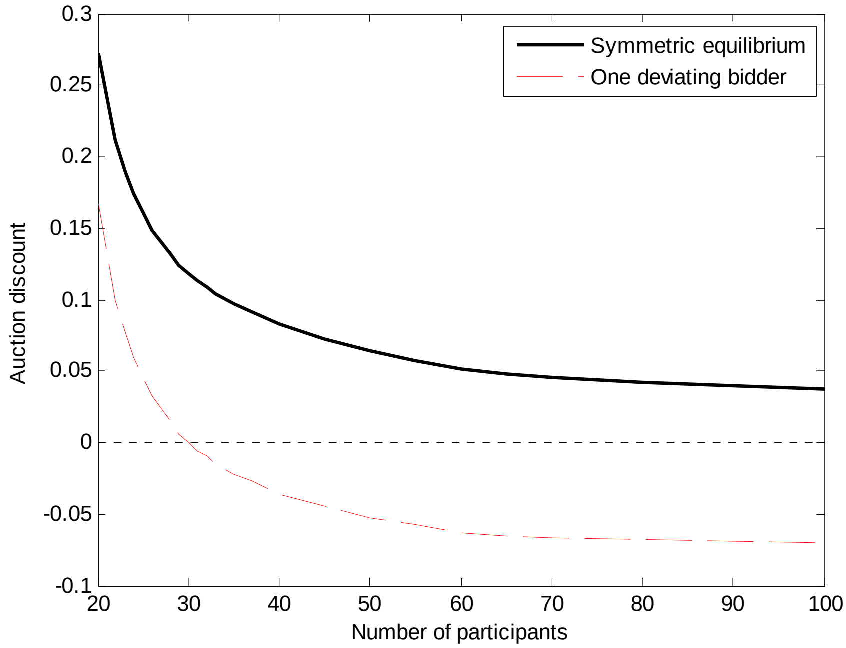


Figure 16: 2_3 Effect of Free Rider one bidder makes a high bid – others do not know that one person is doing ; costly information; almost same as ...



Such deviations are, of course, profitable for the auctioneer -- indeed, the amount of underpricing falls, and overpricing can be expected when the number of participants is sufficiently large, as shown in Figure 17.

Figure 17: 2_4: Auction discount with one deviating agent (others do not know about the deviation)

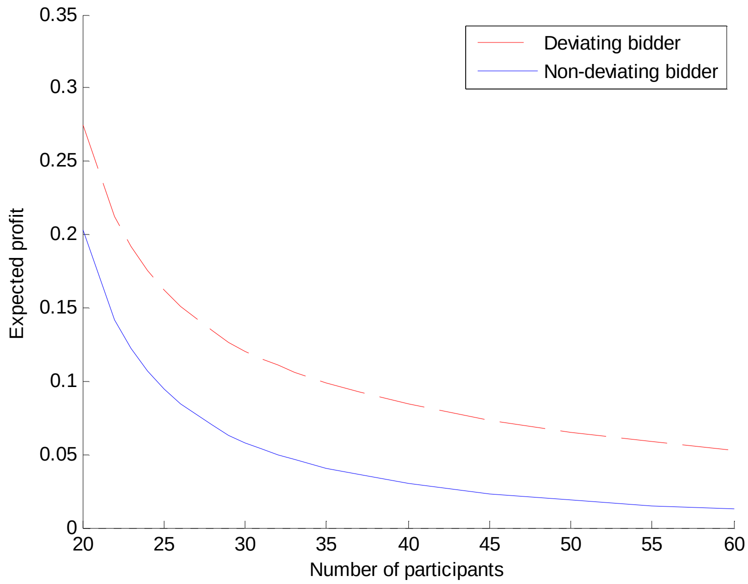


As before, such unexpected deviations cannot persist in equilibrium. If other bidders anticipate this behavior, they would adjust their bids downwards to compensate for the potential losses.

An asymmetric equilibrium

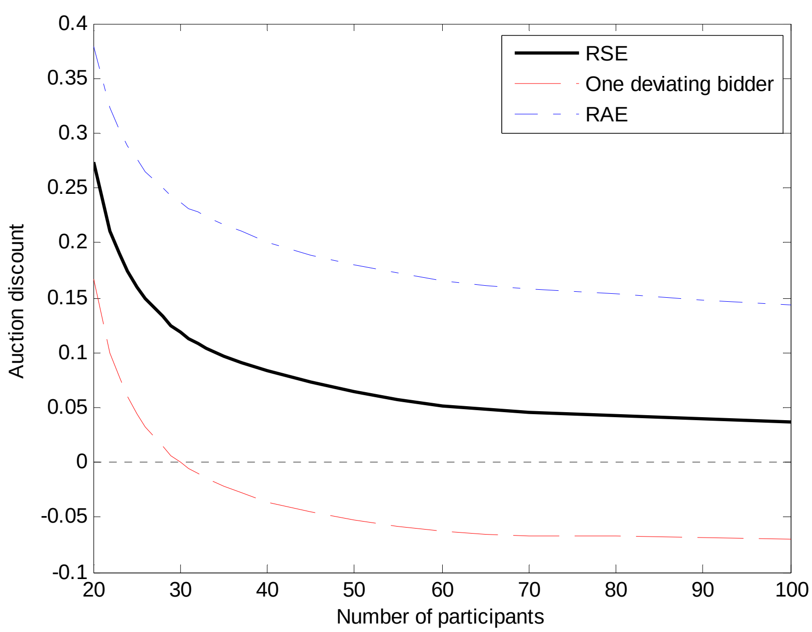
Consider an asymmetric allocation similar to the one discussed above, but where the non-deviating bidders realize that one of them may be playing a different strategy and adjust their bids accordingly. In case of one bidder submitting a "market bid", their problem reduces to the standard Milgrom setup, but with *N-1* bidders competing for *K-1* lots.

Figure 18: 3_1: Bidders' profits in an equilibrium with one deviating bidder



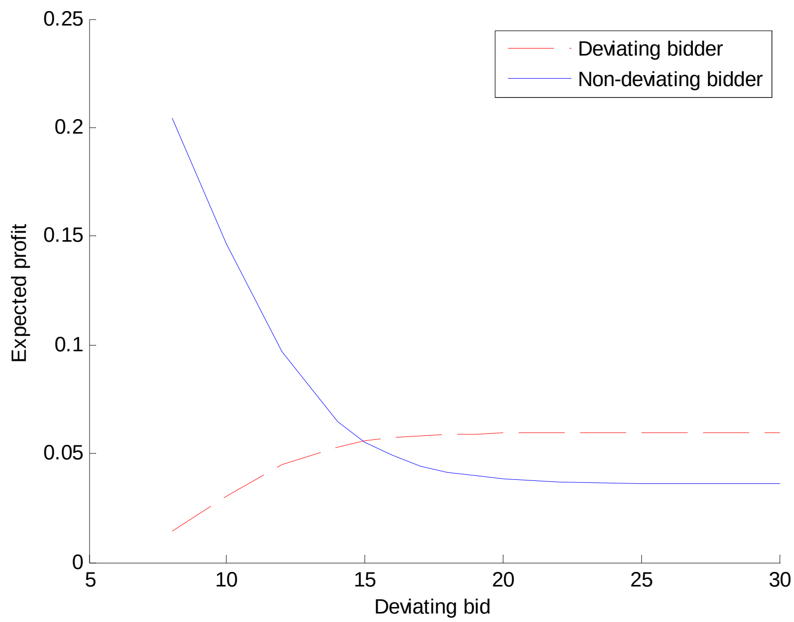
The downward bid adjustment on the part of informed bidders in this equilibrium has a dramatic effect on underpricing: as can be seen from Figure 19, the discount is much larger than either in the symmetric case or, moreover, in the off-equilibrium allocation when the deviation is unexpected.

Figure 19: 3_4: Discount in an asymmetric equilibrium with one deviating agent



Note that the restriction that the uninformed bidders submit "market bids" is not binding in this case: as seen in
Figure 20, the expected return to an uninformed bidder is increasing in the amount bid.

Figure 20: 7_1: Deviating bid and profit of a deviating agent



As the number of deviating bidders increases, so do the profits from deviations (see Figure 21) and the auction discounts (see Figure 22). This happens because such deviations decrease the number of informed bidders, and thereby reduce competition between them.

Figure 21: 6_1: Effect of Free Riders on profits: more than one bidder makes a high bid

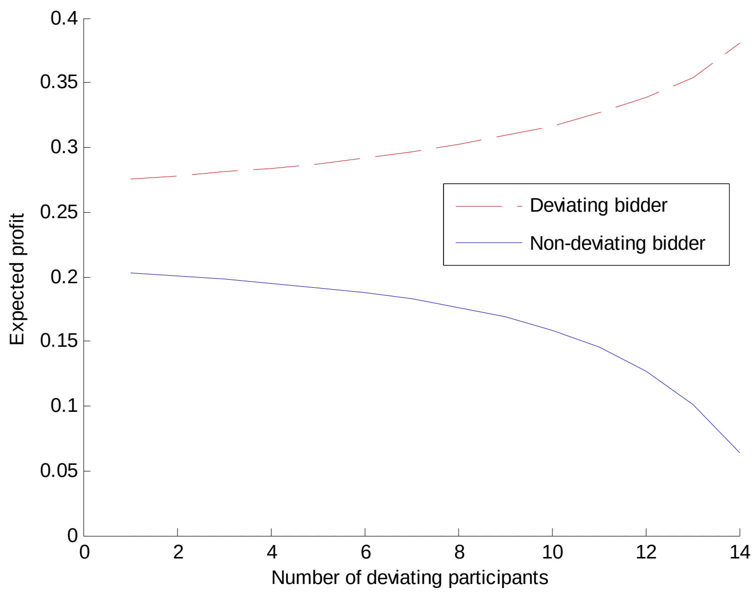
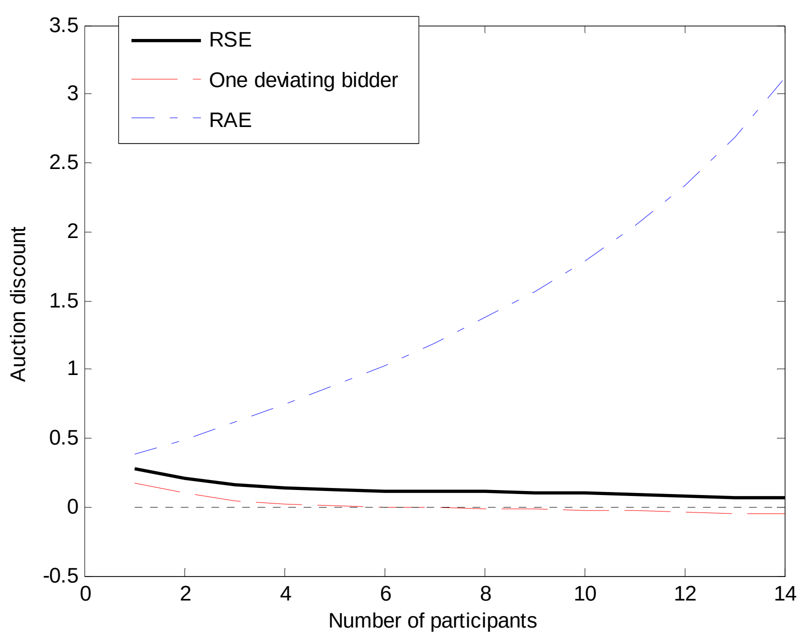


Figure 22: 6_4



It is, however, crucial that $D<K$ for such an equilibrium to exist. When $D \geq K$, there would be no units remaining for the informed bidders, and the "informed bidder" market that determines the price would disappear.

Uncertainty about number of participants

The number of participants in an auction is an important factor in determining the optimal strategy – compare, for example, equilibrium bidding functions corresponding to different values of N in Figure 1. So far, we have always assumed that the number of participants in an auction is fixed. However, in practice it has substantial and this variation substantially affects the bidders' equilibrium strategy.

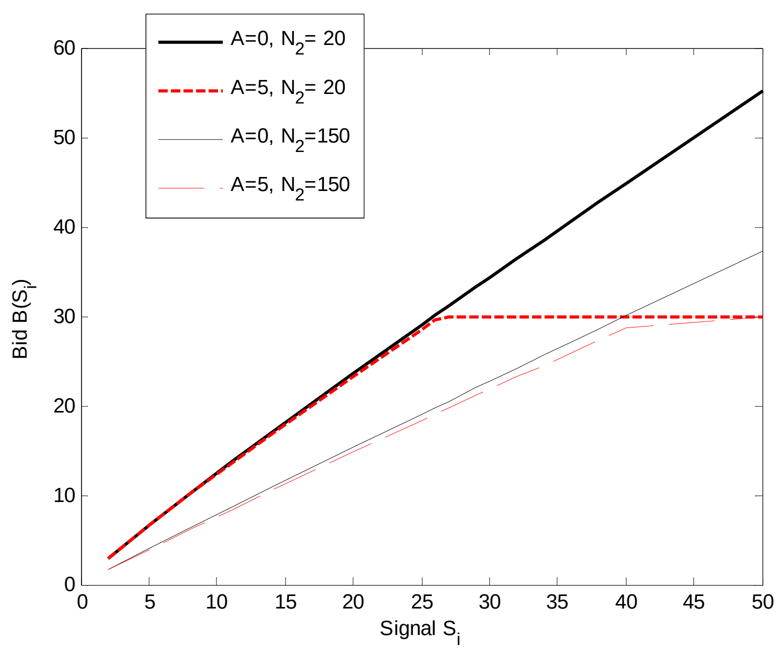
Suppose the number of participants in a given auction N is exogenously determined. There is a pool of N_0 potential participants. The actual number of bidders N can be equal to N_1 with probability ξ and N_2 with probability 1-ξ, where $N_1<N_0$ and $N_2<N_0$. As before, the equilibrium strategy is still given by

$$B_i(s) = \text{argmax } \mathrm{E}\{u(V\text{-}B_i)\geq 0 \mid S_i=s,\ S_{-i}^{(K)}=s, \text{ agent } i \text{ participates}\}$$

with the expectation taken over the joint distribution of V, $\{S_{-i}\}$ and N.

Figure 23 shows the bidding functions of bidders with two different degrees of risk aversion in the deterministic-N case with $N=20$ and in a random-N case with $N_1=20$, $N_2=150$. The number of lots K is the same in both cases, and all other parameters are as in the previous simulations. Note that under constant relative risk aversion the bidders will never bid more than their initial capital.

Figure 23: 13_1: Random number of participants; everyone knows N is random



Effect on underpricing

Figure 24 shows the amount of underpricing in auctions with random number of participants as a function of N_2 (as before, N_1 is kept fixed at 20).

Figure 24: 13_2: Auction discount as a function of N_2

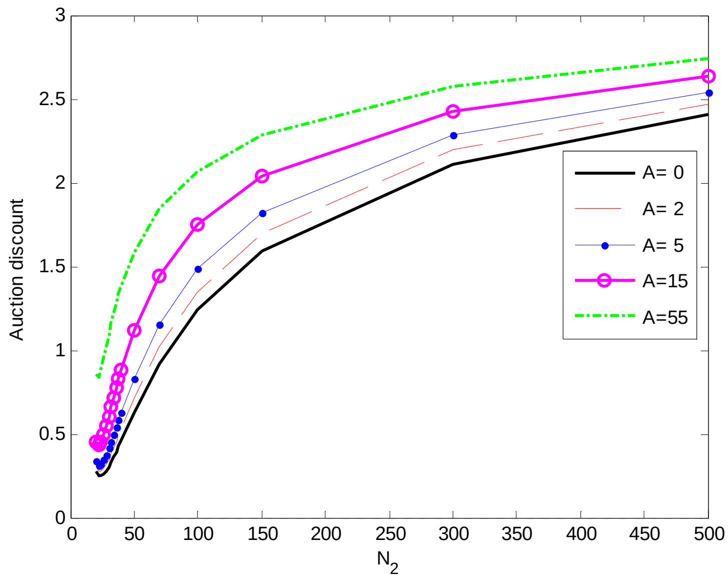


Figure 25 shows auction discount as a function of ξ. The deterministic-*N* case corresponds to the points on the boundary (*N=20* on the left, *N=150* on the right). We can see that with sufficiently high degree of risk-aversion equilibrium auction discount increases substantially – reaching \$3.7 per share, or 37%, with only a 10% chance of a 10x oversubscription, and even under risk-neutrality it can be around 20%.

Figure 25: 14_1: Auction discount as a function of ξ for different values of risk aversion